

Sample Final Exam - MA 1022 B '02

Work all of the problems. The use of calculators and notes is not allowed.

Part I - Basic Skills

Work the following problems and write your answers in the space provided.

1. $\int_0^1 (2x + 1)^5 dx$

$$u = 2x + 1, \quad du = 2dx, \quad x = 0 \text{ implies } u = 1, \quad x = 1 \text{ implies } u = 3.$$

$$\int_0^1 (2x + 1)^5 dx = \frac{1}{2} \int_1^3 u^5 du = \frac{1}{2} \left. \frac{u^6}{6} \right|_1^3 = \frac{3^6}{12} - \frac{1}{12} = \frac{728}{12}.$$

2. $\int \frac{9}{9 + t^2} dt$

$$9 \int \frac{1}{3^2 + t^2} dt = \frac{9}{3} \tan^{-1} \left(\frac{t}{3} \right) + C.$$

3. $\int \frac{\sin(\theta)}{9 + \cos^2(\theta)} d\theta$

$$u = \cos \theta \\ du = -\sin \theta d\theta$$

$$-\int \frac{du}{3^2 + u^2} = -\frac{1}{3} \tan^{-1} \left(\frac{u}{3} \right) + C = -\tan^{-1} \left(\frac{\cos \theta}{3} \right) + C$$

4. $\int_{-1}^1 (x + 1)e^{-2x} dx$

Use integration by parts

$$u = x + 1, \quad du = dx$$

$$dv = e^{-2x} dx, \quad v = -\frac{1}{2}e^{-2x}$$

$$\int_{-1}^1 (x + 1)e^{-2x} dx = (x + 1) \left(-\frac{1}{2}e^{-2x} \right) \Big|_{-1}^1 - \int_{-1}^1 -\frac{1}{2}e^{-2x} \cdot dx \tag{1}$$

$$= 2 \left(-\frac{1}{2}e^{-2} \right) - 0 + \frac{1}{2} \int_{-1}^1 e^{-2x} dx \tag{2}$$

$$= -e^{-2} + \frac{1}{2} \left(-\frac{1}{2}e^{-2x} \right) \Big|_{-1}^1 \tag{3}$$

$$= -e^{-2} - \frac{1}{4} (e^{-2} - e^2) \tag{4}$$

$$= -\frac{5}{4}e^{-2} + \frac{1}{4}e^2. \tag{5}$$

5. $D_x [xe^{-x^2}]$

Use produce rule,

$$D_x [xe^{-x^2}] = 1 \cdot e^{-x^2} + x \cdot (-2x)e^{-x^2} = (1 - 2x^2)e^{-x^2}.$$

6. $\frac{dy}{dx}$ if $y = \arctan(3x)$

Use chain rule $\frac{dy}{dx} = \frac{1}{1 + (3x)^2} \cdot 3 = \frac{3}{1 + 9x^2}$.

7. $D_x \int_x^{x^2} \arctan(3t) dt$

$$\begin{aligned} &= D_x \left(\int_0^{x^2} \arctan(3t) dt + \int_x^0 \arctan(3t) dt \right) \\ &= D_x \left(\int_0^{x^2} \arctan(3t) dt - \int_0^x \arctan(3t) dt \right) \end{aligned}$$

Using the fundamental theorem of calculus and chain rule, we have

$$\arctan(3x^2) \cdot (x^2)' - \arctan(3x)(x)' = \arctan(3x^2) \cdot 2x - \arctan(3x)$$

Part II

Work all of the following problems. Show your work in the space provided. You need not simplify your answers, but remember that on this part of the exam your work and your explanations are graded, not just the final answers.

8. Find the total area of the regions whose boundaries are

$$y = x^3, \quad y = x; \quad 0 \leq x \leq 2.$$

The curve $y = x^3$ is below $y = x$ between 0 and 1 and above it between 1 and 2.

Hence, the area of the region is

$$A = \int_0^1 (x - x^3)dx + \int_1^2 (x^3 - x)dx = \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1 + \left(\frac{x^4}{4} - \frac{x^2}{2} \right) \Big|_1^2 = 2.5.$$

9. Consider the region R whose bounds are given by

$$y = \frac{1}{x}, \quad y = \sqrt{x}; \quad 1 \leq x \leq 4.$$

- (a) Find the volume generated by rotating R about the x -axis.

First find the intersection of the two curves: $y = 1/x, y = \sqrt{x}; \quad 1/x = \sqrt{x}; \quad x = 1$

$$\int_1^4 \pi \left[(\sqrt{x})^2 - \left(\frac{1}{x} \right)^2 \right] dx = \pi \int_1^4 \left(x - \frac{1}{x^2} \right) dx = \frac{27\pi}{4}$$

- (b) Set up the integral for the volume generated when the given region R is rotated about the line $y = 5$. DO NOT BOTHER TO EVALUATE THE INTEGRAL.

$$\pi \int_1^4 \left(5 - \frac{1}{x} \right)^2 - (5 - \sqrt{x})^2 dx$$

10. A large cylindrical tank is filled with maple syrup. The syrup weighs 40 pounds per cubic foot and the tank is 20 feet tall with a base radius of 10 feet. The syrup must be pumped out of the tank to a height of 50 feet above the top of the tank (where there is a large plate containing pancakes). Find the total work done in emptying the tank.

Assume the origin is at the center of the bottom side of the tank. Then work done to lift a layer of maple syrup of thickness Δy at a distance y from the bottom of of tank to 50 feet above the top of the tank is

$$(force)(distance) = \delta (area)(\Delta y)(70 - y) = \delta(\pi 10^2)(\Delta y)(70 - y).$$

Therefore, total work done $W = 40\pi \int_0^{20} 100(70 - y)dy = 4,800,000\pi$.

11. Find the centroid (center of mass) for a quarter circle in the first quadrant (x, y) -plane with with center at $(0,0)$ and radius $r = 2$.

Note: You can simplify your work by using either Pappus' theorem or symmetry. In any case, you must explain your work clearly and justify your answer.

Equation of circle is $x^2 + y^2 = 2^2$ so $f(x) = y = \sqrt{4 - x^2}$

$$\begin{aligned} \bar{x} &= \frac{\int_a^b x[f(x) - g(x)]dx}{\int_a^b [f(x) - g(x)]dx} \\ &= \frac{\int_0^2 x\sqrt{4-x^2}dx}{\frac{1}{4} \cdot \pi \cdot 2^2} \quad (u = 4 - x^2; du = -2xdx; x = 0, u = 4; x = 2, u = 0) \\ &= \frac{\int_4^0 \sqrt{u} \cdot \frac{du}{-2}}{\pi} = \frac{\frac{1}{2} \int_0^4 \sqrt{u} du}{\pi} = \frac{8}{3\pi} \\ \bar{y} &= \frac{\frac{1}{2} \int_a^b [f(x)^2 - g(x)^2]dx}{\int_a^b [f(x) - g(x)]dx} = \frac{\frac{1}{2} \int_0^2 (4 - x^2)dx}{\pi} = \frac{8}{3\pi} \end{aligned}$$

Centroid $(\bar{x}, \bar{y}) = (\frac{8}{3\pi}, \frac{8}{3\pi})$

12. In this problem, you will evaluate the following integral:

$$\int_1^3 4x^2 dx$$

- (a) Construct a Riemann sum with $n = 4$ rectangles to approximate the integral. In particular, define Δx and the points \bar{x}_i where you evaluate the function. You may use a regular mesh (with equal subintervals) in your definition.

$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$

$$\bar{x}_i = 1 + \frac{i}{2}, i = 1, \dots, 4$$

$$\sum_{i=1}^4 4 \left(1 + \frac{i}{2}\right)^2 \frac{1}{2} = \sum_{i=1}^4 2 \left(1 + \frac{i}{2}\right)^2$$

- (b) Step up to the general case: What is the form of the Riemann sum for the example above with an arbitrary number n of subintervals?

$$\Delta x = \frac{2}{n}$$

$$\bar{x}_i = 1 + \frac{2i}{n}$$

$$\sum_{i=1}^n 4 \left(1 + \frac{2i}{n}\right)^2 \frac{2}{n}$$

(c) Compute the limit in the Riemann sum and evaluate the integral.

Note: You may need to use the fact that

$$\sum_{k=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\begin{aligned} \sum_{i=1}^n \frac{8}{n} \left(1 + \frac{2i}{n}\right)^2 &= \sum_{i=1}^n \frac{8}{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) \\ &= \frac{8}{n} \sum_{i=1}^n 1 + \frac{32}{n^2} \sum_{i=1}^n i + \frac{32}{n^3} \sum_{i=1}^n i^2 \\ &= \frac{8}{n} n + \frac{32}{n^2} \frac{n(n+1)}{2} + \frac{32}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= 8 + 16 \left(\frac{n+1}{n}\right) + \frac{16}{3} \frac{(2n^2 + 3n + 1)}{n^2} \\ &= 8 + 16 + \frac{16}{n} + \frac{32}{3} + \frac{16}{n} + \frac{16}{3n^2} \\ \lim_{n \rightarrow \infty} \left(8 + 16 + \frac{16}{n} + \frac{32}{3} + \frac{16}{3n^2}\right) &= 8 + 16 + \frac{32}{3} = \frac{104}{3} \end{aligned}$$

13. Use logarithmic differentiation to find the derivative of $y = x^{\sin(x)}$.

$$\begin{aligned} y &= x^{\sin x} \\ \ln y &= \ln(x^{\sin x}) = \sin x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \cos x \ln x + \sin x \frac{1}{x} \\ \frac{dy}{dx} &= y \left[\cos x \ln x + \frac{\sin x}{x} \right] \\ \frac{dy}{dx} &= x^{\sin x} \left[\cos x \ln x + \frac{\sin x}{x} \right] \end{aligned}$$

14. Suppose that the population P (in millions) of a colony of bacteria is modeled by the differential equation $P'(t) = 0.03P(t)$. If $P(2000) = 250$, find the formula for $P(t)$.

$P'(t) = 0.03P(t)$, then

$$P(t) = Ce^{0.03t}$$

$$P(0) = C = 250.$$

Therefore, $P(t) = 250e^{0.03t}$