

Center of Mass

Purpose

The purpose of this lab is to give you practice with using integrals to determine the centroids of thin plates of uniform thickness and density, but irregular shape.

Background

In designing mechanisms or structures, one often has to deal with distributed forces, that is, forces that do not act at a discrete, finite set of points. The most common example of a distributed force is the force of gravity, which acts on all parts of any body of matter. Other examples are pressure in fluids and electrostatic forces, though there are many others.

One of the basic useful principles of analyzing distributed forces is the idea of replacing them with a single, aggregate force \mathbf{F} that acts at a single point and is somehow equivalent to the original distributed force. This may not always be possible, but this technique has found great use in engineering and science. As a simple example, suppose we have gravity acting on a solid plate of uniform thickness and density, but irregular shape. Finding the equivalent force is really the problem of finding the point where we could exactly balance the plate. This balance point is often called the *center of mass* of the body.

For symmetric objects, the balance point is usually easy to find. For example, the balance point of an empty see-saw is the exact center. Similarly, the balance points for rectangles or circles are just the geometrical centers. For non-symmetric objects, the answer is not so clear, but it turns out that there is a fairly simple algorithm involving integrals for determining balance points.

We begin by restricting our attention to thin plates of uniform density. In Engineering and Science, this type of object is called a lamina. For mathematical purposes, we assume that the lamina is bounded by $x = a$, $x = b$, $y = f(x)$, and $y = g(x)$, with $g(x) \leq f(x)$. Then the book gives the following formulas for the coordinates (\bar{x}, \bar{y}) of the center of mass.

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) dx}{\int_a^b f(x) - g(x) dx}$$
$$\bar{y} = \frac{1}{2} \frac{\int_a^b f(x)^2 - g(x)^2 dx}{\int_a^b f(x) - g(x) dx}$$

We end this section with an example, including Maple commands, for computing the center of mass. Suppose you have a lamina bounded by the curves $f(x) = x^3 - 3x^2 - x + 3$, $g(x) = 3 - x$ and you want to compute the center of mass. To do this in Maple, we first define the two functions.

```
> f := x -> x^3-3*x^2-x+3;  
> g := x -> 3-x;
```

Next you need to find the intersection of the two functions. These intersection points will be used as the limits of integration.

```
> solve(f(x)=g(x),x);
> plot({g(x),f(x)},x=0..3);
```

Notice that the graph of g is above the graph of f . This means that you must switch g and f in the formulas.

Now we are ready to compute the center of mass. Using labels, as shown below, can help you organize your calculations and avoid mistakes. Computing the mass separately also lets you check it. If you get a negative value for the mass, something is wrong and you have to check what you have done. A common mistake is reversing the order of the functions.

```
> mass := int(g(x)-f(x),x=0..3);
> x_bar := int(x*(g(x)-f(x)),x=0..3)/mass;
> y_bar := 1/2*int(g(x)^2-f(x)^2,x=0..3)/mass;
```

Exercises

- Consider the lamina bounded by the curves $f(x) = 1 - x^4$ and $g(x) = \frac{(1 - x^2)}{2}$.
 - Find the intersection points of the two functions.
 - Plot the two functions using the intersection points as the domain.
 - Find the center of mass of the lamina. Does the center of mass lie inside or outside of the lamina?
- Now consider the lamina bounded by the curves $f(x) = 1 - x^4$ and $g(x) = \frac{4(1 - x^2)}{5}$.
 - Find the intersection points of the two functions.
 - Plot the two functions using the intersection points as the domain.
 - Find the center of mass of the lamina. Does the center of mass lie inside or outside of the lamina?
- Finally, consider the lamina bounded by the curves $f(x) = 1 - x^4$ and $g(x) = a(1 - x^2)$ where $0 \leq a \leq 1$.
 - Explain why the x coordinate of the center of mass will be 0 no matter the value of a .
 - Find the y coordinate of the center of mass in terms of a .
 - Determine the values of a for which the center of mass lies inside the lamina. (Hint: At $x = 0$, the bottom edge of the lamina is at $g(0) = a$). Plot $y = a$ and \bar{y} for $0 \leq a \leq 1$ to help you answer the question.