# Exponentials and Logarithms

When the exponential function $f(x) = b^x$ was introduced, (for $b \neq 1, b > 0$) you saw that the function is increasing if $b > 1$ and decreasing if $b < 1$. You can observe the monotonicity by plotting $e^x$ and $0.1^x$.

```
> plot(exp(x),x=-1..1);
> plot(0.1^x,x=-1..1);
```

The logarithmic function $log_b x$ was introduced for $b > 0, b \neq 1$ as the inverse of the exponential. The logarithm is therefore inreasing if $b > 1$ and decreasing if $0 < b < 1$. Indeed you can plot $log_{10} x$ and $log_{0.1} x$.

```
> plot(log[10](x),x=0.1..10);
> plot(log[0.1](x),x=0.1..10);
```

From the monotonicity properties you can compare two logarithms having the same base without computing their values:

$$log_5 10 < log_5 12$$

$$log_{1/2} 9 < log_{1/2} 6$$

From the monotonicity properties you can see that for a fixed $x$ in the interval$(0, 1)$ the logarithm increases with $b$ but for $x$ in the interval $(1, \infty)$ it decreases. As a result you can now compare logarithms with different bases without computing their values.

$$log_3 7 < log_2 7$$

$$log_{1/2} 5 < log_{1/3} 5$$

Here are some examples using Maple to solve logarithmic and exponential equations.

$$5^{x+1} + 5^x + 5^{x-1} = 155$$

$$log_5(2x^2 + 2x + 5) = 2$$

```
> solve(5^(x+1)+5^x+5^(x-1)=155,x);

> solve(log[5](2*x^2+2*x+5)=2,x);
```

# Inverse Functions

Consider the functions $f, g$ defined by

$$f(x) = e^x + e^{-x}, g(x) = 7x + \frac{14}{3}$$

To be able to get an inverse the function must be one-to-one. You can plot the functions to get a hint as to whether they are invertible or not.

```
> f:=x->exp(x)+exp(-x);
> plot(f(x),x=-5..5);
> g:=x->7*x+14/3;
> plot(g(x),x=-10..10);
```

Both satisfy the vertical-line test but $f(x)$ is not invertible since it does not satisfy the horizontal-line test. Indeed $f$ is not one-to-one, for instance $f(0.5) = f(-0.5)$. From the plot it seems that the function $g$ is one-to-one. In order to determine its inverse we solve for x.

```
> solve(g(x)=y,x);
```

Thus:

$$g^{-1}(y) = \frac{y}{7} - \frac{2}{3}$$

```
> ginv:=x->x/7-2/3;
```

Now that x and y have been switched let's look at the plot along with the line $y = x$ to see if our functions seem to make sense.

```
> plot({x,g(x),ginv(x)},x=-10..10,y=-10..10);
```

Notice that the functions are mirrored about the $y = x$ line. Let's check that we have computed the right inverse. By definition the composition of the functions should be the line $y = x$ since an inverse is the reflection about this line.

$$(g \circ g^{-1})(y) = y = (g^{-1} \circ g)(x) = x$$

```
> g(ginv(y));
> ginv(g(x));
```

# Exercises

1. Solve for x in the equations.

   (a)

   $$\ln(x) + x^x = e$$

   (b)

   $$\ln(x) + \ln(x+1) = 2$$

   (c)

   $$3^x + 2^x = 5^x$$

   (d)

   $$log_{10}(100) = \frac{\sqrt{x}}{7}$$

2.

$$g(x) = e^x - 4$$

$$f(x) = 3^x$$

$$h(x) = \frac{ln(x^3 - 1)}{x}$$

**A** Plot the three functions. State whether each function is invertible or not and why.

**B** Find the inverse of the invertible function(s).

**C** Plot each of the functions and its inverse along with the line $y = x$.