Solids of Revolution

Introduction

The purpose of this lab is to use Maple to study solids of revolution. Solids of revolution are created by rotating curves in the x-y plane about an axis, generating a three dimensional object.

Background

So far we have used the integral mainly to to compute areas of plane regions. It turns out that the definite integral can also be used to calculate the volumes of certain types of three-dimensional solids. The class of solids we will consider in this lab are called *Solids of Revolution* because they can be obtained by revolving a plane region about an axis.

As a simple example, consider the graph of the function $f(x) = x^2+1$ for $-2 \leq x \leq 2$.

```
> with(CalcP7):
> f := x -> x^2+1;
> plot(f(x),x=-2..2,y=0..5);
```

If we take the region between the graph and the x-axis and revolve it about the x-axis, we obtain the solid pictured in the next graph.

```
> revolve(f(x),x=-2..2);
```

The `revolve` command has other options that you should read about in the help screen. For example, you can speed the command up by only plotting the surface generated by revolving the curve with the `nocap` argument, and you can also plot a solid of revolution formed by revolving the area between two functions. Try the following examples. (Note: The last example shows how to use `revolve` with a piecewise defined function using the `piecewise` command.)

```
> revolve({f(x),0.5},,x=-2..2,y=-1);
> revolve(cos(x),x=0..4*Pi,y=-2,nocap);
> revolve({5,x^2+1},,x=-2..2);
> g := x-> piecewise(x<0,-x+1/2,x^2-x+1/2);
> revolve(g(x),x=-1..2);
```

It turns out that the volume of the solid obtained by revolving the region between the graph and the x-axis about the x-axis can be determined from the integral

$$\pi \int_{-2}^{2} (x^2+1)^2 \, dx$$

to have the value $\frac{412}{15}\pi$. More generally, if you revolve the area under the graph of $g(x)$ for $a \leq x \leq b$ about the x-axis, the volume is given by

$$\pi \int_{a}^{b} (g(x))^2 \, dx$$

The integral formula given above for the volume of a solid of revolution comes, as usual, from a limit process. Recall the rectangular approximations we used for plane regions. If you think of taking one of the rectangles and revolving it about the x-axis, you get a disk whose radius is the height h of the rectangle and thickness is Δx, the width of the rectangle. The volume of this disk is $\pi h^2 \Delta x$. If you revolve all of the rectangles in the rectangular approximation about the x-axis, you get a solid made up of disks that approximates the volume of the solid of revolution obtained by revolving the plane region about the x-axis.

To help you visualize this approximation of the volume by disks, the `LeftDisk` procedure has been written. The syntax for this command is similar to that for `revolve`, except that the number of subintervals must be specified. The examples below produce approximations with five and ten disks.

```
> LeftDisk(f(x),x=-2..2,5);
> LeftDisk(f(x),x=-2..2,10);
```

Finding Volumes of Revolution

In order to calculate the volume of a solid of revolution, you can either use the `int` command.

```
> Pi*int(f(x)^2,x=-2..2);
> evalf(Pi*int(f(x)^2,x=-2..2));
```

Exercises

1. A sphere of radius 1 is generated when the graph of $\sqrt{1-x^2}$ for $-1 \leq x \leq 1$ is revolved about the x axis. According to the standard formula, the volume of this sphere is $4/3\pi$. Using the `LeftInt` command, determine the minimum number `n` of disks required to approximate the volume of the sphere to within 0.02. Include a plot of these disks with this value of n using the LeftDisk command. Also include a plot of the sphere.

2. Compute the volume of the solid generated by revolving the region bounded by the x-axis and the graph of the function $h(x) = xe^{-x} + 1/2 + \sin(x)$ for $-1 \leq x \leq 4$ about the x-axis.

3. A machining operation begins with a cylindrical blank of radius 1 inch and length 24 inches. After the machining process, the radius of the part is given by $\frac{9}{10} + \frac{\sin(2\pi x/4)}{10}$ where x is distance measured along the axis of the piece. That is, x satisfies $0 \leq x \leq 24$. Compute the volume of the original blank that is removed in the machining process. Include a plot of the original blank and the final piece.