Improper Integrals

Purpose

The purpose of this lab is to use Maple to introduce you to the notion of improper integral and to give you practice with this concept by using it to prove convergence or divergence of integrals involving unbounded integrands or unbounded intervals or both.

Getting Started

To assist you, there is a worksheet associated with this lab that you can open. In Maple go to File - Open and copy the following:

`\\storage\academics\math\calclab\MA1023\Improper_int_start_2017.mws`

Background

Our basic theorem for $\int_a^b f(x)\,dx$ is that the integral exists if $f(x)$ is continuous on the closed interval $[a,b]$. We have actually gone beyond this theorem a few times, and integrated functions that were bounded and had a finite number of jump discontinuities on $[a,b]$. However, we don't have any theory to help us deal with integrals $\int_a^b f(x)\,dx$ involving one or more of the following.

1. Functions $f(x)$, for example rational functions, that have vertical asymptotes in $[a,b]$ (or are not bounded on $[a,b]$).

2. Integrals where the interval $[a,b]$ is unbounded, for example intervals like $[a,\infty)$, $(-\infty,b]$, or $(-\infty,\infty)$.

We have already seen at least one example of the problems you can run into if the function is unbounded. Recall the clearly absurd result

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = -2$$

that is obtained by blindly applying the FTOC. The second type of problem, where the interval of integration is unbounded, occurs often in applications of calculus, such as the Laplace and Fourier transforms used to solve differential equations. It also occurs in testing certain kinds of infinite series for convergence or divergence, as we will learn later.

We start with the following definition.

Definition 1 *We say that the integral*

$$\int_a^b f(x)\,dx$$

is improper if one or both of the following conditions is satisfied.

1. *The interval of integration is unbounded.*

2. *The function $f(x)$ has an infinite discontinuity at some point c in $[a,b]$. That is, $\lim_{x\to c} f(x) = \pm\infty$.*

Unbounded integrands

To see how to handle the problem of an unbounded integrand, we start with the following special cases.

Definition 2 *Suppose that $f(x)$ is continuous on $[a, b)$, but $\lim_{x \to b^-} f(x) = \pm\infty$. Then we define*

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx,$$

provided that the limit on the right-hand side exists and is finite, in which case we say the integral converges and is equal to the value of the limit. If the limit is infinite or doesn't exist, we say the integral diverges or fails to exist and we cannot compute it.

Definition 3 *Suppose that $f(x)$ is continuous on $(a, b]$, but $\lim_{x \to a^+} f(x) = \pm\infty$. Then we define*

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx,$$

provided that the limit on the right-hand side exists and is finite, in which case we say the integral converges and is equal to the value of the limit. If the limit is infinite or doesn't exist, we say the integral diverges or fails to exist and we cannot compute it.

Cases where $f(x)$ has an infinite discontinuity only at an interior point $c, a < c < b$ are handled by writing

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and using the definitions to see if the integrals on the right-hand side exist. If *both* exist then the integral on the left-hand side exists. If *either* of the integrals on the right-hand side diverges, then $\int_a^b f(x)\,dx$ does not exist.

Exercises

1. The gamma function is an example of an improper integral often used to approximate non-integer factorials and is defined below:

   $$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\,dx$$

   Evaluate $\Gamma(t)$ by calculating the improper integral for $t = \{1, 2, 5/2, 3, 7/2, 4, 5\}$ and for each integer value of t, check your answer by calculating $(t-1)!$.

2. Both of the following improper integrals given below do not exist. Show, by calculating a limit, why they do not exist.

   A $f(x) = \sin(x)$, interval $[0, \infty)$.

   B $f(x) = \dfrac{1}{(x^2 - 4)}$, interval $[-2, 2]$.

3. Recall from Calculus II that the volume of a solid of revolution formed by rotating $f(x)$ about the x-axis over the interval $[a, b]$ is

$$\pi \int_a^b f(x)^2 \, dx$$

and its surface area is

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} \, dx$$

A Find the volume of the solid obtained by revolving the curve $y = \dfrac{1}{x}$ about the x-axis, between $x = 1$ and $x = \infty$. Repeat this using $y = \dfrac{1}{x^3}$.

B Plot both functions on the same graph.

C Find the surface area of each solid of revolution.

D Is it possible to have a finite volume but an infinite surface area?