Improper Integrals

Purpose

The purpose of this lab is to use Maple to introduce you to the notion of improper integral and to give you practice with this concept by using it to prove convergence or divergence of integrals involving unbounded integrands or unbounded intervals or both.

Background

We start with the following definition.

Definition 1 *We say that the integral*

$$\int_a^b f(x)\,dx$$

is improper if one or both of the following conditions is satisfied.

1. *The interval of integration is unbounded.*
2. *The function $f(x)$ has an infinite discontinuity at some point c in $[a,b]$. That is, $\lim_{x\to c} f(x) = \pm\infty$.*

Unbounded integrands

To see how to handle the problem of an unbounded integrand, we start with the following special cases.

Definition 2 *Suppose that $f(x)$ is continuous on $[a,b)$, but $\lim_{x\to b^-} f(x) = \pm\infty$. Then we define*

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx,$$

provided that the limit on the right-hand side exists and is finite, in which case we say the integral converges and is equal to the value of the limit. If the limit is infinite or doesn't exist, we say the integral diverges or fails to exist and we cannot compute it.

Definition 3 *Suppose that $f(x)$ is continuous on $(a,b]$, but $\lim_{x\to a^+} f(x) = \pm\infty$. Then we define*

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx,$$

provided that the limit on the right-hand side exists and is finite, in which case we say the integral converges and is equal to the value of the limit. If the limit is infinite or doesn't exist, we say the integral diverges or fails to exist and we cannot compute it.

Cases where $f(x)$ has an infinite discontinuity only at an interior point $c, a < c < b$ are handled by writing

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

and using the definitions to see if the integrals on the right-hand side exist. If *both* exist then the integral on the left-hand side exists. If *either* of the integrals on the right-hand side diverges, then $\int_a^b f(x)\,dx$ does not exist.

Examples Here is a simple example using Maple to show that $\displaystyle\int_0^2 \frac{1}{x}\,dx$ doesn't exist.

```
> ex1 := int(1/x,x=a..2);
```

```
> limit(ex1,a=0,right);
```

The example above used the `right` option to `limit` because the right-hand limit was needed. If you need a left-hand limit, use the `left` option in the `limit` command. Maple can usually do the limit within the `int` command.

```
> int(1/x,x=0..2);
```

Unbounded intervals of integration

These are handled in a similar fashion by using limits. The definition we need the most is given below.

Definition 4 *Suppose $f(x)$ is continuous on the unbounded interval $[a, \infty)$. Then we define*

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

provided the limit on the right-hand side exists and is finite, in which case we say the integral converges and and is equal to the value of the limit. If the limit is infinite or fails to exist we say the integral diverges or fails to exist.

The other two cases are handled similarly. You are asked to provide suitable definitions for them in one of the exercises.

Examples Using the definition for $\displaystyle\int_2^\infty \frac{1}{x^2}$.

```
> ex2:=int(1/x^2,x=2..a);
> limit(ex2,a=infinity);
```

This command shows that Maple takes the limit definition into account in the `int` command.

```
> int(1/x^2,x=2..infinity);
```

Exercises

1. The gamma function is an example of an improper integral often used to approximate non-integer factorials and is defined below:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$$

   Evaluate $\Gamma(t)$ by calculating the improper integral for $t = \{1, 2, 5/2, 3, 7/2, 4, 5\}$ and for each integer value of t, check your answer by calculating $(t-1)!$.

2. Both of the following improper integrals given below do not exist. Show, by calculating a limit, why they do not exist.

   A $f(x) = \sin(x)$, interval $[0, \infty)$.

   B $f(x) = \dfrac{1}{(x^2-4)}$, interval $[-2, 2]$.

3. Plot $\frac{1}{x}$ and $\frac{1}{x^3}$ on a single graph.Recall from Calculus II that the volume of a solid of revolution formed by rotating $f(x)$ about the x-axis over the interval $[a, b]$ is

$$\pi \int_a^b f(x)^2\, dx$$

   . Find the volume of the solid obtained by revolving the curve $y = \dfrac{1}{x}$ about the x-axis, between $x = 1$ and $x = \infty$. Repeat this using $y = \dfrac{1}{x^3}$.