Power Series

Purpose

The purpose of this lab is to learn how to determine convergence of series using Maple as well as introduce you to power series and their radius of convergence.

Background

An infinite series is the sum of an infinite sequence. More precisely, the sum of an infinite series is defined as the limit of the sequence of the partial sums of the terms of the series a_n, provided this limit exists. If no finite limit exists, then we say that the series is divergent.

The sum of an infinite series is defined as $\lim_{n\to\infty} S_n$, where S_n is the partial sum of the first n terms of the series. However, because of the algebraic difficulty of expressing S_n as a function of n, it is usually not possible to find sums by directly using the definition.

So, if we can generally not work from the definition, what can be done? There are several convergence tests that provide us with some needed tools. These are tests that tell us if a series converges, but in the case that the series does converge, does not tell us the sum of the series.

The nth-Term Test

The series

$$\sum_{n=1}^{\infty} a_n$$

diverges if $\lim_{n\to\infty} a_n$ is non-zero or does not exist.

The Comparison Test

For **Direct Comparison**, consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

1. If $a_n < b_n$ for $n < N$, where N is some integer and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.

2. If $a_n > b_n$ for $n < N$, where N is some integer and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also.

For **Limit Comparison**, consider the same series above
$IF \lim_{n\to\infty} \dfrac{a_n}{b_n} = L$

1. And if $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

2. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges also.

3. If $L \to \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also.

The Integral Test

The Integral Test for convergence is a method used to test convergence of an infinite series of nonnegative terms. The series

$$\sum_{n=1}^{\infty} a_n$$

converges if and only if the integral

$$\int_1^{\infty} f(x)\,dx$$

is finite, where $f(x)$ is a positive, non-increasing and continuous function defined on the interval $[1, \infty)$ and $f(n) = a_n$ an for all n.

The Ratio Test

The Ratio Test for convergence of a series can be thought of as a measurement of how fast the series is increasing or decreasing. This can be found by looking at the ratio $\frac{a_{n+1}}{a_n}$ as $n \to \infty$. Given the series $\sum_{n=0}^{\infty} a_n$, suppose that

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$$

Then

1. the series converges if $L < 1$,
2. the series diverges if $L > 1$,
3. the test is inconclusive if $L = 1$.

Root Test

Given the series $\sum_{n=0}^{\infty} a_n$, suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$$

Then

1. the series converges absolutely if $\rho < 1$,
2. the series diverges if $\rho > 1$,
3. the test is inconclusive if $\rho = 1$.

Examples

To find if the series $\sum_{n=1}^{\infty} \frac{1}{(n^4+4)^{1/3}}$ converges or diverges the integral test will be used. First plot the function a_n to see if it satisfies the necessary conditions.

```
>f:=x->1/((x^4+4)^(1/3));
>plot(f(x),x=1..100);
```

The function is positive, non-increasing, and continuous therefore the integral test can be used.

```
>evalf(int(f(x),x=1..infinity));
```

Since the integral is a finite number the **series** converges.

To find if the series $\sum_{n=1}^{\infty} \frac{1.01^n}{n^{100}}$ converges or diverges the root test will be used.

```
>limit(abs((1.01^n)/(n^100))^(1/n),n=infinity);
```

Since the limit is greater than 1, the **series** diverges.

Power Series

A power series about $x = 0$ has the form

$$\sum_{n=0}^{\infty} a_n x^n$$

and a power series about $x = a$ has the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where a_n are the constant coefficients of the powers of x.

Radius or Interval of Convergence

The radius of convergence of a power series can usually be found by using the Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

Next, you would need to solve for the interval of x values such that $|L| < 1$. Remember that power series always converge:

1. If L exists and is non-zero, the power series converges absolutely on some interval $|x - a| < R$.
2. If $L = 0$, the power series converges for all x.
3. If $L \to \infty$, the power series converges only at $x = 0$

Example

Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{n*(x+3)^n}{5^n}$. Using the ratio test:

```
>a:=n->(n*(x+3)^n)/(5^n);
>L:=limit(abs(a(n+1)/a(n)),n=infinity);
>solve(L<1,x);
```

The interval of convergence is a length from -8 to 2 which is 10, therefore the **radius** of convergence is half that or 5.

Exercises

1. Use any convergence test to determine if each of the following series converges or diverges and explain your answer.

   (a)
   $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{(1-\frac{1}{n})^n}{2^n}$$

   (c)
   $$\sum_{n=1}^{\infty} \frac{\ln(n^2+1)}{n\sqrt{n}}$$

   (d)
   $$\sum_{n=1}^{\infty} \frac{n^3}{e^{n^3}}$$

   (e)
   $$\sum_{n=1}^{\infty} (\sqrt{n+\sqrt{n}}-\sqrt{n})$$

   (f)
   $$\sum_{n=1}^{\infty} \frac{n!2^n}{n^n}$$

   (g)
   $$\sum_{n=1}^{\infty} \frac{1.99^n}{n^{100}(2-\frac{1}{n})^n}$$

2. Use Maple to find the radius and interval of convergence for the each of the following power series

   (a)

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n}$$

   (b)

$$\sum_{n=1}^{\infty} n! x^n$$

   (c)

$$\sum_{n=1}^{\infty} 12n^2(x+12)^n$$