TOPICS

- More Approximation
  - $\hookrightarrow$ Taylor Polynomials
  - $\hookrightarrow$ Estimating Error
- Re-Quiz #1-B
  - $\hookrightarrow$ L'Hopital's Rule

Remember?

You can almost always approximate a function with a polynomial.

The basic idea:

Pick the constants in the polynomial so that you match the function value and as many derivatives as you want...

.... at a fixed point.

Start with an Example:

From Last Time...

$$\cos(\theta) \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$$

How much error do you make if you use the polynomial to ...

- approximate $\cos(1)$? .... $\cos(1) \approx 13/24$?
- approximate $\int_0^1 \cos(\theta) d\theta$?

How does your calculator compute cos anyway?

Some Numbers:

$$\cos(\theta) \approx 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4$$

θ	$\cos(\theta)$	$P_4(\theta)$	Error
0.0	1	1	0
0.2	0.9801	0.9801	8.9×10^{-6}
1.0	0.5403	0.5417	1.4×10^{-3}
2.0	-0.4164	-0.3333	8.2×10^{-2}
10.0	-0.8391	367.667	$3.7 \times 10^{+2}$

Main Question:

Can you estimate the error if you cannot compute the function?

Main Answer: YES

You can (almost) always estimate the error...

$$|\cos(\theta) - P_4(\theta)| = \left|\frac{f^{(6)}(\zeta)}{6!}\right| \theta^6 \leq \frac{1}{6!}\theta^6$$

where $0 \leq \zeta \leq \theta$

θ	Actual Error	Estimated Error
0.0	0	0
0.2	8.9×10^{-6}	8.9×10^{-6}
1.0	1.4×10^{-3}	1.4×10^{-3}
2.0	8.2×10^{-2}	8.92×10^{-2}
10.0	378	$1,389$

ASIDE

The cosine is a little strange...there are no odd terms!

MAIN MESSAGE:

$$
\begin{aligned}
f(x) &= f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 \\
&\quad + R_2(x)
\end{aligned}
$$

where

$$R_2(x) = \left|\frac{f^{(3)}(\zeta)}{3!}\right| \theta^3$$

The error depends on the *next term.*

This is called "Taylor's Theorem with Remainder."

GENERAL CASE:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}\cdot(x-a)^2 \\ &\quad + \frac{f'''(a)}{3!}\cdot(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}\cdot(x-a)^n \end{aligned}$$

The error depends on the *next term:*

$$R_n(x) = \left|\frac{f^{n+1}(\zeta)}{(n+1)!}\right| \theta^{n+1}$$

You cannot *compute* this, but you can *estimate* it!

Example:

How many terms do you need in order to estimate $\cos(1)$ with an error of at most 0.0000001?

Answer:

$$\text{Error} \le \frac{1}{(n+1)!}1^{n+1} = \frac{1}{(n+1)!} \le 0.0000001$$

You need $(n+1)! \ge 10,000,000$

($n = 10$ gives you $11! =$ 39,916,800)

ANNOUNCEMENTS:

- Homework #3 Due Wednesday in Conference

  Sec. 10.1: 7, 13, 25, 26

  Sec. 10.2: 9, 17, 19

- Quiz #3: Thursday in Lecture (Improper Integrals)

- Evening Help Sessions:

  Tuesday and Thursday, 7:00–9:00pm in SH106

- Section 4 "in the box" tomorrow