TOPICS

- More Approximation
  - $\hookrightarrow$ Estimating Error
- Numerical Integration
  - $\hookrightarrow$ Trapezoids
  - $\hookrightarrow$ Parabolas

Picture:

Improving on Riemann...

Look at One Trapezoid:

Area in the trapezoid = (width) × (average height)

$$A_1 = h \cdot \left[\frac{f(x_0) + f(x_1)}{2} \right]$$

Picture It:

Area under one zoid...interval $[0, h]$

Linear Interpolation:

Hit the endpoints:

$$\ell(x) := f(0) + \frac{f(h) - f(0)}{h - 0} \cdot (x - 0)$$

Integrate it:

$$\begin{aligned}
\int_0^h \ell(x)dx &= \int_0^h \left(f(0) + \frac{f(h) - f(0)}{h - 0} \cdot x\right)dx \\
&= f(0) \cdot h + \frac{f(h) - f(0)}{h - 0} \cdot \frac{h^2}{2} \\
&= \frac{h}{2}\left[f(h) + f(0)\right]
\end{aligned}$$

Error in Linear Interpolation:

$$f(x) - \ell(x) = \frac{f''(\xi)}{2}x(x - h)$$

Error in one trapezoid:

$$\begin{aligned}
\int_0^h (f(x) - \ell(x))\, dx &= \int_0^h \frac{f''(\xi)}{2}x(x - h)dx \\
&= \frac{f''(\xi)}{2} \int_0^h x(x - h)dx \\
&=
\end{aligned}$$

Sum the Errors:

You have

$$n = \frac{(b-a)}{h}$$

little trapezoids.

$$\text{Error } = -\frac{(b-a)^3}{12n^2} f''(\zeta)$$

Summary: Trapezoidal Rule

Divide the interval $[a, b]$ into n equal subintervals...

$$\int_a^b f(x)dx \approx \frac{h}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\Big]$$

Call the approximation T_n.

Main Theorem

$$\int_a^b f(x)dx = T_n + E_n$$

where

$$T_n = \frac{h}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\Big]$$

and

$$E_n = -\frac{(b-a)^3}{12n^2}f''(c)$$

where $a \leq c \leq b$

Look at the Error:

$$E_n = -\frac{(b-a)^3}{12n^2}f''(c)$$

- Error increases as $(b-a)$ increases
- Large concavity (f'') is bad.
- If you double the number of intervals, you divide the error by ____________
- You can *estimate* the error if you have

$$M = \max\Big(f''(x)\Big)$$

on the interval.

Example:

$$\int_{-1}^{-1} e^{-x^2}\,dx = \operatorname{erf}(1) \cdot \sqrt{\pi}$$

Look at the Error:

Parabolas Instead...

Picture:

Answer: $\int_0^1 f(x)dx = \frac{1}{6}f(0) + \frac{2}{3}f(1/2) + \frac{1}{6}f(1)$

Sneaky Derivation:

You are looking for an approximation formula:

$$\int_0^1 f(x)dx = A_0 f(0) + A_1 f(1/2) + A_2 f(1)$$

It should be *exact* for

- constants
- lines
- parabolas

Make it so.

Making It So:

$$f(x) \equiv 1 \qquad 1 = A_0 + A_1 + A_2$$

$$f(x) = x \qquad 1 = \frac{1}{2}A_1 + A_2$$

$$f(x) = x^2 \qquad 1 = \frac{1}{4}A_1 + A_2$$

Solve for the constants:

$$A_0 = \frac{1}{6}, \quad A_1 = \frac{2}{3}, \quad A_2 = \frac{1}{6}$$

Simpson's Rule:

Apply your interpolatin formula on a bunch of subintervals...

Example #1:

Example #2:

ANNOUNCEMENTS:

- Homework #3 Due Wednesday in Conference

  Sec. 10.1: 7, 13, 25, 26

  Sec. 10.2: 9, 17, 19

- Quiz #2: Thursday in Lecture (Improper Integrals)