TOPICS

- More Solving Equations... Approximately
  - ↪ Newton's Error
  - ↪ Fixed Point Method
- Beginning Convergence
  - ↪ Sequences of Numbers

Recall Newton... Picture:

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Question:

How fast does the method converge?

Example:

Approximating $\sqrt{2}$ by solving $x_2 - 2 = 0$

You *double* the number of correct digits at each step.

Step	x_n	Correct Digits
0	3	0
1	1.183	1
2	1.4621	2
3	1.4149984	4
4	1.4142137800	7
∞	1.4142135623	**

Remark:

Recursion builds a *sequence*

$$x_0, x_1, x_2, \ldots x_n, x_{n+1}, \ldots$$

and you hope that

$$\lim_{n \to \infty} x_n = \sqrt{2}$$

Theorem:

Newton is always this fast.

$$\left| x_{n+1} - r \right| \leq \frac{M}{2m} \left(x_n - r \right)^2$$

where

$$f'(x) \geq m > 0 \quad \text{and} \quad \left| f''(x) \right| \leq M$$

and r is a root for $f(x) = 0$.

Justification: Taylor!

$$0 = f(r) = f(x_n) + f'(x_n)(r - x_n) + \frac{f''(c)}{2} \cdot (r - x_n)^2$$

Rearrange terms:

$$x_n - \frac{f(x_n)}{f'(x_n)} - r = \frac{f''(c)}{2f'(x_n)}(x_n - r)^2$$

Fixed Point Methods

Example: You can solve $\cos(\theta) = \theta$ with one finger

1. Put your calculator in radian mode
2. start anywhere (call it θ_0)
3. Hit the cos key until you get tired

WHY Does it Work?

You are building a sequence

$$\theta_0, \theta_1, \theta_2, \ldots \theta_n, \ldots$$

and you hope that

$$\theta^\star = \lim_{n \to \infty} \theta_n = \cos(\theta^\star)$$

Awfully Optimistic!

Aside:

IF there is a limit, it has to be a fixed point:

$$\begin{array}{ccc} \theta_{n+1} & = & \cos(\theta_n) \\ \downarrow & & \downarrow \\ \theta^\star & = & \cos(\theta^\star) \end{array}$$

Here is Why:

Look at the difference:

$$\begin{aligned} \theta_{n+1} - \theta_n &= f(\theta_n) - f(\theta_{n-1}) \\ &= f'(\zeta) \cdot \Big(\theta_n - \theta_{n-1}\Big) \\ &< p \cdot \Big(\theta_n - \theta_{n-1}\Big) \end{aligned}$$

IF the derivative is always less than $p < 1$.

Error always decreases: This is the Key!

Quick Summary:

1. Newton and Bisection will solve $f(x) = 0$
2. Fixed point method will solve $f(x) = x$
3. All methods are recursive (and easy to program).

New Stuff:

How do you PROVE that

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \ ?$$

Intuition:

When n is big, $1/n^2$ is small.

Mathematics:

For any number $\epsilon > 0$, there exists a $N > 0$ such that

$$n > N \quad \text{guarantees} \quad \left|\frac{1}{n^2} - 0\right| < \epsilon$$

Definition of Convergence

Say that

$$\lim_{n\to\infty} a_n = L$$

if for any number $\epsilon > 0$, there exists a $N > 0$ such that

$$\left|a_n - L\right| < \epsilon \quad \text{whenever} \quad n > N.$$

Apply the Definition

$$\lim_{n\to\infty} \frac{1}{n^2} = 0$$

Find a formula for N in terms of ϵ.

Example:

Does $a_n = 2 + (0.99)^n$ converge?

Two Key Limits:

#1: $\lim_{n \to \infty} \frac{1}{n} = 0$

#2: $\lim_{n \to \infty} p^n = 0$ if and only if $-1 < p < 1$

Many Key Theorems

ANNOUNCEMENTS:

- Homework #4 Due Tomorrow in Lecture

  Sec. 10.3: 7, 10, 13, 17;

  Sec. 10.4: 8, 15, 18, 19
- Quiz #4: Tomorrow in Lecture: Solving Equations
- Make-up for ALL Quizzes: Friday (Sign up Wednesday)

  2:00, 2:30, 3:00, 3:30

  in Stratton Hall 106