Lecture 4  3/20/06
Applications of
Improper integrals
Fourier transform
signal processing,
heat transfer,
Laplace transform
signal processing
control theory
solving ODE's
Probability
Lab
Sequences and Series

How does your calculator calculate $e^x, \sin(x), \ln(x)$?

*Uses Taylor series.*

**Sequences**

An infinite sequence is a list of numbers $a_1, a_2, a_3, a_4, \ldots$

*Example*

1, 2, 3, 4, 5, \ldots

1, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

1, -1, 1, -1, 1, \ldots
Rules for sequences

1. formula
\[ a_n = \frac{1}{n}, \quad n = 1, 2, 3, 4, \ldots \]

2. pattern
\[ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \]

3. recursion
\[ a_1 = 1 \]
\[ a_{n+1} = \frac{1}{2} a_n, \quad n = 1, 2, 3, \ldots \]
gives
\[ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \]
Notation: $\{a_n\}$ shorthand for $a_1, a_2, a_3, a_4, \ldots$

Def: The sequence $\{a_n\}$ converges to a real number $L$ or has limit $L$ and we write

$$\lim_{n \to \infty} a_n = L$$

if for each $\varepsilon > 0$ there is a positive integer $N$ such that $|a_n - L| < \varepsilon$ for each $n \geq N$. A sequence that fails to converge is said to diverge.
**Thm 4:**

If \( a_n = f(n) \), \( n = 1, 2, 3, \ldots \), then

\[
\lim_{x \to 00} f(x) = L \quad \text{then} \quad \lim_{n \to 00} a_n = L
\]

**Ex**

\[
a_n = \frac{n}{n+1}
\]

\[
f(x) = \frac{x}{x+1}
\]

\[
\lim_{x \to 00} \frac{x}{x+1} = \lim_{x \to 00} \frac{1}{1} = 1
\]

\[
\infty \quad \text{form}
\]

so

\[
\lim_{n \to 00} \frac{n}{n+1} = 1
\]
\[ a_n = \frac{n}{e^n} \]

\[ \lim_{n \to \infty} \frac{n}{e^n} = \lim_{n \to \infty} \frac{1}{e^n} = 0 \]

**\text{Thm 2}:** If \( \lim_{n \to \infty} a_n = A \)

and \( f \) is continuous at \( x = a \) then

\[ \lim_{n \to \infty} f(a_n) = f(A) \]
\[ \lim_{n \to \infty} \left( \frac{n}{2n+3} \right)^{3/2} \]

\[ = \left( \lim_{n \to \infty} \frac{n}{2n+3} \right)^{3/2} \]

\[ = \left( \frac{1}{2} \right)^{3/2} \]

**Thm 1**: Limit laws for sequences

If \( \lim a_n = A \), \( \lim b_n = B \), then

1. \( \lim ca_n = cA \), \( c \) any constant
2. \( \lim (a_n + b_n) = A + B \)
3. \( \lim (a_n b_n) = AB \)
4) \( \lim_{n \to \infty} \left( \frac{a_n}{b_n} \right) = \frac{A}{B} \) if \( B \neq 0 \)

\[ \text{Ex} \]
\[ \lim_{n \to \infty} \frac{n+3}{2n+5} = \frac{\lim_{n \to \infty} n+3}{\lim_{n \to \infty} 2n+5} \]

Theorem doesn't help since
\[ \lim_{n \to \infty} n+3 = \infty, \lim_{n \to \infty} 2n+5 = \infty \]
\[ a_n = \frac{(-1)^n}{n} \]

\[-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\]

can't use Thm 4, since
\[ \left(\frac{-1}{x}\right)^x \]
doesn't make sense.

\[ \text{Thm 3 (squeeze law)} \]
If \( a_n \leq b_n \leq c_n \) \( n = 1, 2, 3, \ldots \)
and if \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \)
then \( \lim_{n \to \infty} b_n = L \).
\[ \frac{1}{n} \leq (-1)^n \leq 1 \]

so

\[ -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \]

Know \( \lim_{n \to \infty} \frac{1}{n} = 0 \)

and \( \lim_{n \to \infty} -\frac{1}{n} = 0 \)

so \( \lim_{n \to \infty} \frac{(-1)^n}{n} = 0 \)

by the squeeze law.
Ex \[ a_n = \frac{\sin(n)}{n+1} \]

\[-1 \leq \sin(n) \leq 1 \]

so

\[-\frac{1}{n+1} \leq \frac{\sin(n)}{n+1} \leq \frac{1}{n+1} \]

\[\lim_{n \to \infty} \frac{1}{n+1} = \lim_{n \to \infty} \frac{-1}{n+1} = 0\]

so \[\lim_{n \to \infty} \frac{\sin(n)}{n+1} = 0\]

Ex \[ a_n = \frac{n \sin(n)}{n+1} \]

\[-\frac{n}{n+1} \leq \frac{n \sin(n)}{n+1} \leq \frac{n}{n+1}\]
\[
\lim_{n \to \infty} \frac{n}{n+1} = 1, \quad \lim_{n \to \infty} \frac{-n}{n+1} = -1
\]

limits aren't equal so squeeze law doesn't apply.
Ex

suppose \( |r| < 1 \) (or \(-1 < r < 1\))

then \( \lim_{n \to \infty} r^n = 0 \)

\(-|r|^n \leq r^n \leq |r|^n \)

so assume \( 0 < r < 1 \)

Let \( \varepsilon > 0 \) be given. Can we find \( N \) such that \( n \geq N \)

means \( r^n < \varepsilon \)

\( \ln(r^n) < \ln(\varepsilon) \)

or \( n \ln(r) < \ln(\varepsilon) \)

\( n > \frac{\ln(\varepsilon)}{\ln(r)} \)

\( 0 < r < 1 \) means \( \ln(r) < 0 \)