Operations on power series

Purpose

The purpose of this lab is to acquaint you with several useful techniques for working with power series. These techniques include substitution and multiplication and division of series by simple polynomials.

Getting Started

To assist you, there is a worksheet associated with this lab that contains examples and even solutions to some of the exercises. You can open this worksheet after you start up Maple by choosing Open... from the File menu and then typing the following file name.

\storage.wpi.edu\academics\math\calclab\MA1023\Powseries_start.mw

You should read through the lab before you load this worksheet into Maple. Once you have read to the exercises, start up Maple, load the worksheet Powseries_start.mw, and go through it carefully by reading the text and running the commands. Then you can start working on the exercises.

Background

The general form of a power series in \( x - a \) is given below.

\[
f(x) = \sum_{k=0}^{\infty} a_k (x - a)^k
\]

The number \( a \) is called the base point of the power series. In this lab, we will consider only the special case \( a = 0 \). Historically, power series have been used most often to approximate functions that do not have simple formulas. Two examples that should be familiar from class are given below.

The most familiar example of a power series is the geometric series.

\[
\frac{1}{1 - x} = \sum_{k=0}^{\infty} x^k, \text{ for } |x| < 1
\]

Another example is the power series for \( \exp(x) \) (or \( e^x \)), which is

\[
\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \text{ for all } x
\]

The exercises also use the series for \( \cos(x) \) and \( \sin(x) \), which you haven’t seen yet, but will very soon.

Once you have a convergent series representation for a function, it can be manipulated in several ways to generate convergent power series representations of related functions. The rest of the Background describes several different techniques for generating power series representations of functions that are related to power series that are already known. The four techniques are substitution, multiplication and division, integration, and differentiation.
Substitution

Suppose you have a power series representation for \( f(x) \), and you want a power series representation for \( f(ax^p) \) where \( a \) is a constant and \( p \) is a positive integer. The theory for power series says that you can obtain the power series you want simply by substitution. For example, to get a power series for \( \exp(2x) \) you just take the series for \( \exp(x) \) and substitute \( 2x \) for \( x \), obtaining

\[
\exp(2x) = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}.
\]

Writing out the first few terms gives

\[
\exp(2x) = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots
\]

If you use such a substitution, you have to be careful if the series is only valid for a finite interval. For example, suppose you wanted to find the power series for

\[
f(t) = \frac{1}{1 + 2t}
\]

You can obtain the desired series by substitution as

\[
\frac{1}{1 + 2t} = \sum_{k=0}^{\infty} (-1)^k (2t)^k
\]

but you have to be careful because this formula is not valid for all values of \( t \). In fact this formula is only valid if \( |t| < 1/2 \). The reason for this is that the series for \( 1/(1 - x) \) is only valid if \( |x| < 1 \) and when we substitute \(-2t\) for \( x \), the formula only makes sense if \( |2t| < 1 \).

Multiplication and Division

If you have a power series representation for \( f(x) \), and you want the power series for something like \( x^2 f(x) \), you can just multiply each term of the series for \( f(x) \) by \( x^2 \). If the leading term for the power series representation of \( g(x) \) is \( x^k \) for some integer \( k > 0 \), you can use division to obtain the power series representation for \( g(x)/x^n \) for any integer \( n \leq k \). Some examples follow.

\[
\frac{x}{1 - x} = x + x^2 + x^3 + \ldots
\]

\[
x^2 \exp(x) = x^2 + x^3 + \frac{x^4}{2} + \ldots
\]

\[
\frac{\exp(x) - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots
\]
Term-by-term integration and differentiation

One of the nice properties of power series is that they can be integrated and differentiated term-by-term. For example,

\[
\frac{d}{dx} \frac{1}{1-x} = \frac{1}{(1-x)^2}
\]

so we have the following power series representation.

\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots, \text{ for } |x| < 1
\]

Note that the interval of convergence is exactly the same as for the original series, neglecting the behavior at the endpoints.

Integrating a power series term-by-term is very similar, but you may have to include a constant of integration. For example, integrating the power series representation for \(\exp(x)\) term by term gives

\[
\int 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \, dx = C + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

You would have to set \(C = 1\) to make the right hand side the power series representation for \(\exp(x)\).

Exercises

1. For the following functions, compare the partial sums of power series with base points \(a = 0\) up to various orders obtained directly with those obtained by applying addition, subtraction, substitution, multiplication, or division to the power series for \(\exp(x)\), \(\cos(x)\), \(\sin(x)\), or the geometric series for \(1/(1-x)\). See the examples in the Getting Started worksheet.
   
   (a) \(f(x) = \exp(2x)\).
   
   (b) \(f(x) = x \exp(-x^2)\).
   
   (c) \(f(x) = (1 - \cos(x))/x^2\)
   
   (d) \(f(x) = x^2/(1-x)\)
   
   (e) \(f(x) = 1 - \cos(3x)\)

2. Use integration followed by substitution to generate the first three terms in the power series with base point \(a = 0\) for \(\ln(1+x^2)\). Start with the series for \(1/(1+x)\).

3. You know that the derivative of \(e^{-x}\) is \(-e^{-x}\). Is it true that if you differentiate the power series with base point \(a = 0\) for \(e^{-x}\) you get the power series for \(-e^{-x}\)? Investigate this by differentiating the power series for \(e^{-x}\) and comparing it to the power series for \(-e^{-x}\). Use at least five terms in your comparison.