MID-TERM EXAM SOLUTIONS

In working the following, use no calculators, books, or notes except the formula sheet provided. Note the number of points for each problem. Show your work.

1. (8 points) Find the distance from the origin \((0, 0, 0)\) to the midpoint of the line segment joining \((1, 2, 3)\) and \((3, 4, 5)\).

Solution: The midpoint is \(\left(\frac{1+3}{2}, \frac{2+4}{2}, \frac{3+5}{2}\right) = (2, 3, 4)\). The distance is
\[
\sqrt{(2-0)^2 + (3-0)^2 + (4-0)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}.
\]

2. (12 points) Find the equation of the plane through \((1, 0, 0)\), \((0, 1, 0)\), and \((1, 1, 1)\).

Solution: First, find a normal to the plane: Two vectors parallel to the plane are \(\langle 0 - 1, 1 - 0, 0 - 0 \rangle = \langle -1, 1, 0 \rangle\) and \(\langle 1 - 1, 1 - 0, 1 - 0 \rangle = \langle 0, 1, 1 \rangle\). Then a normal vector is \(\mathbf{n} = \langle -1, 1, 0 \rangle \times \langle 0, 1, 1 \rangle = \langle 1, -1, -1 \rangle\). Having \(\mathbf{n}\) and choosing \((1, 0, 0)\) as a point in the plane, we have the equation \(1 \cdot (x - 1) + 1 \cdot (y - 0) - 1 \cdot (z - 0) = 0\), or \(x + y - z = 1\).

3. (12 points, 3 points each) Which of the following are true? (Here, \(\mathbf{u}\), \(\mathbf{v}\), and \(\mathbf{w}\) are vectors in 3-space and \(\mathbf{i}\), \(\mathbf{j}\), and \(\mathbf{k}\) are the standard 3-space basis vectors.)
   a. \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})\)
      False: The right-hand side is a vector crossed with a scalar, which doesn’t make sense.
   b. \((\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{j}\) True: \((\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{k} \times \mathbf{i} = \mathbf{j}\).
   c. \(\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = 0\) True: \(\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i} = 0\).
   d. If \(\mathbf{u} \cdot \mathbf{v} \neq 0\), then the angle between \(\mathbf{u}\) and \(\mathbf{v}\) is \(\tan^{-1}\left(\frac{|\mathbf{u} \times \mathbf{v}|}{\mathbf{u} \cdot \mathbf{v}}\right)\).
      True: If \(\theta\) is the angle between \(\mathbf{u}\) and \(\mathbf{v}\), then \(\frac{|\mathbf{u} \times \mathbf{v}|}{\mathbf{u} \cdot \mathbf{v}} = \frac{|\mathbf{u}||\mathbf{v}| \sin \theta}{|\mathbf{u}||\mathbf{v}| \cos \theta} = \tan \theta\)
4. Suppose the position of a particle moving along a curve in 3-space is given by \( \mathbf{r}(t) = \mathbf{i} + t \mathbf{j} + \frac{1}{2}t^2 \mathbf{k} \).

a. (20 points) Find the velocity \( \mathbf{v} \), the acceleration \( \mathbf{a} \), and the speed \( |\mathbf{v}| \) as functions of \( t \).

Solution: \( \mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{j} + t \mathbf{k} \), \( \mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{k} \), and \( |\mathbf{v}(t)| = \sqrt{1 + t^2} \).

b. (20 points) Find the tangential and normal components of acceleration \( a_T \) and \( a_N \) as functions of \( t \).

Solution:

\[
a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(\mathbf{j} + t \mathbf{k}) \cdot \mathbf{k}}{\sqrt{1 + t^2}} = \frac{t}{\sqrt{1 + t^2}} \\
a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = \sqrt{1 - \frac{t^2}{1 + t^2}} = \frac{1}{\sqrt{1 + t^2}}
\]

c. (6 points) Find the curvature \( \kappa \) at \( t = 0 \).

Hint: Use the formula \( \kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \) evaluated at \( t = 0 \).

Solution:

\[
\kappa = \frac{|\mathbf{v}(0) \times \mathbf{a}(0)|}{|\mathbf{v}(0)|^3} = \frac{|\mathbf{j} \times \mathbf{k}|}{(1)^3} = 1
\]

5. (12 points, 4 points each)

a. Change \( z^2 + y^2 + z^2 = 9 \) to cylindrical coordinates.

Solution: We have \( z^2 + y^2 = r^2 \), so the equation becomes \( r^2 + z^2 = 9 \).

b. Change \( z^2 + y^2 + z^2 = 9 \) to spherical coordinates.

Solution: We have \( z^2 + y^2 + z^2 = \rho^2 \), so this becomes \( \rho^2 = 9 \), or \( \rho = 3 \).

c. Change \( \rho = 2 \cos \phi \) to cylindrical coordinates. Hint: Multiply both sides by \( \rho \).

Solution: Multiplying both sides by \( \rho \) gives \( \rho^2 = 2 \rho \cos \phi \). We have \( \rho^2 = z^2 + y^2 + z^2 = r^2 + z^2 \) and \( z = \rho \cos \phi \), so the equation becomes \( r^2 + z^2 = 2z \).

6. (10 points, 5 points each) Suppose \( f(x, y) = x + e^{-xy} \).

a. Find \( f_x(x, y) \) and \( f_y(x, y) \).

Solution: \( f_x(x, y) = 1 - ye^{-xy} \) and \( f_y(x, y) = -xe^{-xy} \).

b. Find \( f_{xy}(x, y) \).

Solution:

\[
f_{xy}(x, y) = \frac{\partial}{\partial y} (f_x(x, y)) = \frac{\partial}{\partial y} (1 - ye^{-xy}) = -e^{-xy} + ye^{-xy} = (xy - 1)e^{-xy}
\]