MA 1024 Lab 4: Partial derivatives, directional derivatives, and the gradient

Purpose

The purpose of this lab is to acquaint you with using Maple to compute partial derivatives, directional derivatives, and the gradient.

Getting Started

To assist you, there is a worksheet associated with this lab that contains examples. You can copy that worksheet to your home directory with the following command. On your Maple screen File-Open. In the white rectangle copy:

```
\\storage\academics\math\calclab\MA1024\Pardiff_grad_start.mw
```

You can copy the worksheet now, but you should read through the lab before you load it into Maple. Once you have read to the exercises, start up Maple, load the worksheet `Pardiff_start.mw`, and go through it carefully. Then you can start working on the exercises.

Background

For a function $f(x)$ of a single real variable, the derivative $f'(x)$ gives information on whether the graph of f is increasing or decreasing. Finding where the derivative is zero was important in finding extreme values. For a function $F(x, y)$ of two (or more) variables, the situation is more complicated.

Partial derivatives

A differentiable function, $F(x, y)$, of two variables has two partial derivatives: $\partial F/\partial x$ and $\partial F/\partial y$. As you have learned in class, computing partial derivatives is very much like computing regular derivatives. The main difference is that when you are computing $\partial F/\partial x$, you must treat the variable y as if it was a constant and vice-versa when computing $\partial F/\partial y$.

The Maple commands for computing partial derivatives are `D` and `diff`. The **Getting Started** worksheet has examples of how to use these commands to compute partial derivatives.

Directional derivatives

The partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ of F can be thought of as the rate of change of F in the direction parallel to the x and y axes, respectively. The directional derivative $D_{\mathbf{u}}F(\mathbf{p})$, where $\mathbf{u}$ is a unit vector, is the rate of change of F in the direction $\mathbf{u}$. There are several different ways that the directional derivative can be computed. The method most often used for hand calculation relies on the gradient, which will be described below. It is also possible to simply use the definition

$$D_{\mathbf{u}}F(\mathbf{p}) = \lim_{h \to 0} \frac{F(\mathbf{p} + h\mathbf{u}) - F(\mathbf{p})}{h}$$

to compute the directional derivative. However, the following computation, based on the definition, is often simpler to use.

$$D_{\mathbf{u}}F(\mathbf{p}) = \left. \frac{d}{dt} F(\mathbf{p} + t\mathbf{u}) \right|_{t=0}$$

One way to think about this that can be helpful in understanding directional derivatives is to realize that $\mathbf{p} + t\mathbf{u}$ is a straight line in the x, y plane. The plane perpendicular to the x, y plane that contains this straight line intersects the surface $z = F(x, y)$ in a curve whose z coordinate is $F(\mathbf{p} + t\mathbf{u})$. The derivative of $F(\mathbf{p} + t\mathbf{u})$ at $t = 0$ is the rate of change of F at the point $\mathbf{p}$ moving in the direction $\mathbf{u}$.

Maple doesn't have a simple command for computing directional derivatives. There is a command in the `tensor` package that can be used, but it is a little confusing unless you know something about tensors. Fortunately, the method described above and the method using the gradient described below are both easy to implement in Maple. Examples are given in the `Getting Started` worksheet.

The Gradient

The gradient of F, written ∇F, is most easily computed as

$$\nabla F(\mathbf{p}) = \frac{\partial F}{\partial x}(\mathbf{p})\mathbf{i} + \frac{\partial F}{\partial y}(\mathbf{p})\mathbf{j}$$

As described in the text, the gradient has several important properties, including the following.

- The gradient can be used to compute the directional derivative as follows.

$$D_{\mathbf{u}}F(\mathbf{p}) = \nabla F(\mathbf{p}) \cdot \mathbf{u}$$

- The gradient $\nabla F(\mathbf{p})$ points in the direction of maximum increase of the value of F at $\mathbf{p}$.
- The gradient $\nabla F(\mathbf{p})$ is perpendicular to the level curve of F that passes through the point $\mathbf{p}$.
- The gradient can be easily generalized to apply to functions of three or more variables.

Maple has a fairly simple command `grad` in the `linalg` package (which we used for curve computations). Examples of computing gradients, using the gradient to compute directional derivatives, and plotting the gradient field are all in the `Getting Started` worksheet.

Exercises

1. For the function $f(x, y) = \sqrt{1 + x^2 + y^2} \cos(xy)$,

   A) Generate a surface plot of f over the domain $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

   B) Generate a contouplot over the same domain.

2. Using method 2 from the `Getting Started` worksheet compute the directional derivative of f at the point $(-1, \frac{\pi}{2})$ in each of the directions below. Explain your results in terms of being positive, negative or zero and what that tells about the surface at that point in the given direction.

   (a) $\mathbf{u} = \langle \cos(\pi/4), \sin(\pi/4) \rangle$

   (b) $\mathbf{u} = \langle -1, 1 \rangle$

   (c) $\mathbf{u} = \langle 2, \pi \rangle$

3. Using the method from the `Getting Started` worksheet, plot the gradient field and the contours of f on the same plot. Describe the surface of f at the origin using both the gradient field and the countour plot in your explanation. You should probably use a smaller domain like $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$ to help you see what is going on at $(0, 0)$. After your explanation, plot the surface of f again over this smaller domain and rotate your plot until it is in a position that supports your claim of what is happening at the origin.