Partial Derivatives and their Geometric Interpretation

Purpose

The purpose of this lab is to acquaint you with using Maple to compute partial derivatives.

Background

For a function $f(x)$ of a single real variable, the derivative $f'(x)$ gives information on whether the graph of f is increasing or decreasing. For a function $F(x, y)$ of two (or more) variables, you need to specify which independent variable is being derived.

Partial derivatives

A differentiable function, $F(x, y)$, of two variables has two partial derivatives: $\partial F/\partial x$ and $\partial F/\partial y$. As you have learned in class, computing partial derivatives is very much like computing regular derivatives. The main difference is that when you are computing $\partial F/\partial x$, you must treat the variable y as if it was a constant and vice-versa when computing $\partial F/\partial y$.

The Maple commands for computing partial derivatives are D and `diff`. The `diff` command can be used on both expressions and functions whereas the D command can be used only on functions. The examples below show all first order and second order partials in Maple.

```
> f := (x,y) -> x^2*y^2-x*y;
> diff(f(x,y),x);
> diff(f(x,y),y,y);
> D[1](f)(x,y);
> D[1,2](f)(x,y);
```

Note in the above D command that the **1** in the square brackets means **x** and the **2** means **y**. The next example shows how to evaluate the mixed partial derivative of the function given above at the point $(-7, 8)$.

```
> subs({x=-7,y=8},diff(f(x,y),x,y));
> D[1,2](f)(-7,8);
```

Tangent Lines and Planes

The tangent plane like the tangent line to a single variable function is based on derivatives, however the partial derivatives are used for the tangent plane. Let's start with the equation of the tangent line to the function $f(x)$ at the point where $x = a$. Recall, the general equation of a line at the point $(a, f(a))$ having slope m is $y - f(a) = m(x - a)$. This can be rewritten knowing that the derivative is the slope of a tangent line as $y = f'(a)(x - a) + f(a)$. Similarly for a funcion of two variables, the equation of the plane tangent to

$z = f(x, y)$ at the point (a, b) has the equation $z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$. The following examples will show you how the line and the plane can easily be translated to Maple syntax.

```
> g := x-> sin(x)-x^3/7+x^2;
> tl := D(g)(5)*(x-5)+g(5);
> plot({g(x),tl},x=-2..8);
```

The next example shows how to find the tangent plane to the function $j(x, y) = \dfrac{1}{1 + x^2 + y^2}$ at $(1/8, 1/4, z)$. You could write the partials with `diff` or D. This example uses D as it is easier to plug in the the point with this syntax; with `diff` the `subs` command would be used.

```
> j:=(x,y)->1/(1+x^2+y^2);
> tp:=D[1](j)(1/8,1/4)*(x-1/8)+D[2](j)(1/8,1/4)*(y-1/4)+j(1/8,1/4);
> plot3d([j(x,y),tp],x=-1..1,y=-1..1,style=patchnogrid);
```

To find a point where the tangent plane is horizontal, you would need to solve where both first order partials are equal to zero simultaneously.

```
> solve({diff(f(x,y),x)=0,diff(f(x,y),y)=0},{x,y});
```

There are two plot commands for three-dimensional graphs **plot3d** and **implicitplot3d**. The first assumes the $=\mathbf{z}$ and is therefore not included in the command.

```
>plot3d(g(x,y),x=-10..10,y=-10..10,axes=boxed);
```

Note the difference in the syntax for the second plot command. An equal sign must be included in the equation. This gives the flexibility of being able to graph equations without having to solve for z first.

```
>with(plots):
>implicitplot3d([x^2+y^2=1-z^2,x=y],x=-1.1..1.1,y=-1.1..1.1,z=-1.1..1.1,
axes=normal,color=[black,magenta],style=[wireframe,patchnogrid],thickness=2);
```

Three-dimensional plots have many options. Some have been used inthe above command. To see more information try these two commands.

```
>?plot,colornames
>?plot3d,options
```

Exercises

1. Compute the three distinct second order partial derivatives of

$$f(x, y) = \cos(2(y - 1)) + \sin(x - y)$$

   at the point $(\frac{\pi}{6}, \frac{-\pi}{6})$ using the **diff** command and then again using the **D** command.

2. Using the same function

   a) Plot the function and the plane $x = \frac{\pi}{2}$ on the same graph. Use intervals $-3 \le x \le 3$, $-3 \le y \le 3$, $-3 \le z \le 3$.

   b) Graph the two-dimensional intersection of the plane $x = \frac{\pi}{2}$ and $f(x, y)$.

   c) Does your two-dimensional graph look like the intersection from your three-dimensional graph? Be sure to use the same ranges to properly compare and rotate the 3-D graph.

   d) Find the derivative of $f(x, y)$ in the $x = \frac{\pi}{2}$ plane using first the **diff** command and then the **D** command.

3. Find the equation of the plane tangent to the surface $z = \dfrac{x \sin(x + y)}{\sqrt{4 + x^2 + y^2}}$ at the point $(-1, 2)$. Plot both the tangent plane and the surface on the same graph over the intervals $-3 \le x \le 1$ and $0 \le y \le 4$. Be sure to rotate the graph to see that the planes are tangent to the surface.