MA 1024 Lab: Local and Global Extrema

Purpose

The purpose of this lab is to acquaint you with techniques for finding local and global extreme values of functions of two variables.

Background

Many applications of calculus involve finding the maximum and minimum values of functions. For example, suppose that there is a network of electrical power generating stations, each with its own cost for producing power, with the cost per unit of power at each station changing with the amount of power it generates. An important problem for the network operators is to determine how much power each station should generate to minimize the total cost of generating a given amount of power.

Classifying Local Extrema

In single-variable calculus, we found that we could locate candidates for local extreme values by finding points where the first derivative vanishes. For functions of two dimensions, the condition is that both first order partial derivatives must vanish at a local extreme value candidate point. Such a point is called a stationary point. It is also one of the three types of points called critical points.

Finding and classifying the stationary points of a function $f(x, y)$ requires several steps. First, the partial derivatives must be computed. Then the stationary points must be solved for by solving where both partial derivatives equal zero simultaneously.

$$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$$

Next, one must check for the presence of singular points, which might also be local extreme values. Finally, each critical point must be classified as a local maximum, local minimum, or a saddle point using the **second-partials test**:

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 > 0$ and $f_{xx}(x_0, y_0) > 0$ then $f(x_0, y_0)$ is a local minimum.

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 > 0$ and $f_{xx}(x_0, y_0) < 0$ then $f(x_0, y_0)$ is a local maximum.

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 < 0$ then $f(x_0, y_0)$ is a saddle point.

If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2 = 0$ then no conclusion can be made.

Locating Global Extrema

In one-dimensional calculus, the absolute or global extreme values of a function occur either at a point where the derivative is zero, a boundary point, or where the derivative fails to exist. The situation for a function of two variables is very similar, but the problem is much more difficult because the boundary now consists of curves instead of just endpoints of intervals. Consider the function:

```
>f := (x, y) -> (50*x - 50)*y*exp(-(x - 1/2)^2 - (y + 1/2)^2);
>plot3d(f(x, y), x = -1 .. 1, y = -1 .. 1, style = patchnogrid, numpoints = 12000,
 color = blue);
```

On the circular domain $(x - 1/2)^2 + y^2 = 1/4$

```
>plot([sqrt(-x^2 + x), -sqrt(-x^2 + x)], x = -1 .. 1, y = -1 .. 1);
>with(plots):
>a := plot3d(f(x, y), x = -1 .. 1, y = -1 .. 1, style = patchnogrid,
numpoints = 12000, color = blue):
>b := implicitplot3d((x - 1/2)^2 + y^2 = 1/4, x = -1 .. 1, y = -1 .. 1,
 z = -10 .. 30, style = wireframe, numpoints = 13000):
>display(a, b);
```

As in previous labs you can find the stationary points by solving the partials equal to zero.

```
>cp := solve({diff(f(x, y), x) = 0, diff(f(x, y), y) = 0}, {x, y});
```

The critical points are (1,0) and (0,1/2). The others are not in the domain. But now you need to work along the circular boundary which consists of two functions:$y = \sqrt{-x^2 + x}$ and $y = -\sqrt{-x^2 + x}$. Since they equal y you substitute the y out and take the derivative just like Calculus I.

```
>cp1 := fsolve(diff(f(x, sqrt(-x^2 + x)), x) = 0, x);
>cp1y := subs(x = cp1, sqrt(-x^2 + x));
>cp2 := fsolve(diff(f(x, -sqrt(-x^2 + x)), x) = 0, x);
>cp2y := subs(x = cp2, -sqrt(-x^2 + x));
```

Finally you must consider where the boundary equations intersect. The two semicircles intersect at the points (0,0) and (1,0). Calculating the z-values:

```
>f(cp1, cp1y)
>f(cp2, cp2y)
>f(0, 0)
>f(1, 0)
>f(0, 0.5)
```

The absolute maximum is (0.30,-0.46, 15.38) and the absolute minimum is (0, 0.5, -7.16)

Exercises

1. For each three-dimensional function, graph it on the domain $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$ using `plot3d` and then `contourplot`. Finally, find and classify all critical points as a local maximum, local minimum or saddle point using the second derivative test. Use syntax that you have learned from previous labs.

   (a) $f(x) = xy$

(b) $g(x) = \frac{x+y}{1+x^2+y^2}$

2. For the function $h(x, y) = cos(x)cos(y)$, graph it on the domain $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$ using `plot3d` and then `contourplot`. On the contourplot, how are the critical points shown? The maximums and minimums? The saddle points?

3. For the function $f(x, y) = x^2 + 2yx - 2y^2 + x - 2y$, find the absolute extrema on the triangular domain $y = 3$, $x = 3$, $y = -x$.