MA 1024 Lab 4: Partial derivatives, directional derivatives, and the gradient

Purpose

The purpose of this lab is to acquaint you with using Maple to compute partial derivatives, directional derivatives, and the gradient.

Getting Started

To assist you, there is a worksheet associated with this lab that contains examples.On Maple go to File - Open and copy the following into the white rectangle:

```
\\storage\academics\math\calclab\MA1024\Pardiff_grad_start.mws
```

Remember to immediately save it in your own home directory. Once you've copied and saved the worksheet, read through the background on the internet and the background of the worksheet before starting the exercises.

Background

For a function $f(x)$ of a single real variable, the derivative $f'(x)$ gives information on whether the graph of f is increasing or decreasing. Finding where the derivative is zero was important in finding extreme values. For a function $F(x, y)$ of two (or more) variables, the situation is more complicated.

Partial derivatives

A differentiable function, $F(x, y)$, of two variables has two partial derivatives: $\partial F/\partial x$ and $\partial F/\partial y$. As you have learned in class, computing partial derivatives is very much like computing regular derivatives. The main difference is that when you are computing $\partial F/\partial x$, you must treat the variable y as if it was a constant and vice-versa when computing $\partial F/\partial y$.

The Maple commands for computing partial derivatives are `D` and `diff`. The **Getting Started** worksheet has examples of how to use these commands to compute partial derivatives.

Directional derivatives

The partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ of F can be thought of as the rate of change of F in the direction parallel to the x and y axes, respectively. The directional derivative $D_{\mathbf{u}}F(\mathbf{p})$, where $\mathbf{u}$ is a unit vector, is the rate of change of F in the direction $\mathbf{u}$. There are several different ways that the directional derivative can be computed. The method most often used for hand calculation relies on the gradient, which will be described below. It is also possible to simply use the definition

$$D_{\mathbf{u}}F(\mathbf{p}) = \lim_{h \to 0} \frac{F(\mathbf{p} + h\mathbf{u}) - F(\mathbf{p})}{h}$$

to compute the directional derivative. However, the following computation, based on the definition, is often simpler to use.

$$D_{\mathbf{u}}F(\mathbf{p}) = \left. \frac{d}{dt} F(\mathbf{p} + t\mathbf{u}) \right|_{t=0}$$

One way to think about this that can be helpful in understanding directional derivatives is to realize that $\mathbf{p} + t\mathbf{u}$ is a straight line in the x, y plane. The plane perpendicular to the x, y plane that contains this straight line intersects the surface $z = F(x, y)$ in a curve whose z coordinate is $F(\mathbf{p} + t\mathbf{u})$. The derivative of $F(\mathbf{p} + t\mathbf{u})$ at $t = 0$ is the rate of change of F at the point $\mathbf{p}$ moving in the direction $\mathbf{u}$.

Maple doesn't have a simple command for computing directional derivatives. There is a command in the `tensor` package that can be used, but it is a little confusing unless you know something about tensors. Fortunately, the method described above and the method using the gradient described below are both easy to implement in Maple. Examples are given in the `Getting Started` worksheet.

The Gradient

The gradient of F, written ∇F, is most easily computed as

$$\nabla F(\mathbf{p}) = \frac{\partial F}{\partial x}(\mathbf{p})\mathbf{i} + \frac{\partial F}{\partial y}(\mathbf{p})\mathbf{j}$$

As described in the text, the gradient has several important properties, including the following.

- The gradient can be used to compute the directional derivative as follows.

$$D_{\mathbf{u}}F(\mathbf{p}) = \nabla F(\mathbf{p}) \cdot \mathbf{u}$$

- The gradient $\nabla F(\mathbf{p})$ points in the direction of maximum increase of the value of F at $\mathbf{p}$.
- The gradient $\nabla F(\mathbf{p})$ is perpendicular to the level curve of F that passes through the point $\mathbf{p}$.
- The gradient can be easily generalized to apply to functions of three or more variables.

Maple has a fairly simple command `grad` in the `linalg` package (which we used for curve computations). Examples of computing gradients, using the gradient to compute directional derivatives, and plotting the gradient field are all in the `Getting Started` worksheet.

Exercises

For the function $f(x, y) = (x^2 - y^2) * exp(-x^2 - y^2)$,

1. Using either method from the `Getting Started` worksheet, compute the directional derivative of f at the point $(-1/2, 0)$ in each of the directions below. Explain your results in terms of being positive, negative or zero.

   A) $\mathbf{u} = \langle \frac{-4}{5}, \frac{3}{5} \rangle$

   B) $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

   C) $\mathbf{u} = \langle 0, 1 \rangle$

2. A) Repeat the above exercise using the point $(-1, 0)$ in the same directions.

   B) Repeat the above exercise using the point $(0, 1)$ in the same directions.

   C) Repeat the above exercise using the point $(0, 0)$ in the same directions.

   D) What do your results suggest about the surface at these points and how is the result different from the point $(\frac{-1}{2}, 0)$ Could you have come to a conclusion with only two of the above directions? Why?

3. Using the method from the `Getting Started` worksheet, plot the gradient field and the contours of f on the same plot over the intervals $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.On the contourplot, use 30 contours. On the gradient plot use a $[30, 30]$ grid and `fieldstrength=fixed`. Describe the surface of f at each point in exercise 2 using both the gradient field and the countour plot in your explanation.