Partial Derivatives and their Geometric Interpretation

Purpose

The purpose of this lab is to acquaint you with using Maple to compute partial derivatives.

Background

For a function $f(x)$ of a single real variable, the derivative $f'(x)$ gives information on whether the graph of f is increasing or decreasing. For a function $F(x, y)$ of two (or more) variables,you need to specify which independent variable is being derived.

Partial derivatives

A differentiable function, $F(x, y)$, of two variables has two partial derivatives: $\partial F/\partial x$ and $\partial F/\partial y$. As you have learned in class, computing partial derivatives is very much like computing regular derivatives. The main difference is that when you are computing $\partial F/\partial x$, you must treat the variable y as if it was a constant and vice-versa when computing $\partial F/\partial y$.

The Maple commands for computing partial derivatives are `D` and `diff`. The `diff` command can be used on both expressions and functions whereas the `D` command can be used only on functions. The examples below show all first order and second order partials in Maple.

```
> f := (x,y) -> x^2*y^2-x*y;
> diff(f(x,y),x);
> diff(f(x,y),y,y);
> D[1](f)(x,y);
> D[1,2](f)(x,y);
```

Note in the above D command that the **1** in the square brackets means **x** and the **2** means **y**. The next example shows how to evaluate the mixed partial derivative of the function given above at the point $(-7, 8)$.

```
> subs({x=-7,y=8},diff(f(x,y),x,y));
> D[1,2](f)(-7,8);
```

Partial derivatives for implicitly defined expressions

There is an `implicitdiff` command. The important thing to remember is **the second argument** is the variable for the **dependent** variable. As an example, find $\frac{dy}{dz}$ of $xy^2 + e^y = z + y$:

```
>jo := x*y^2 + exp(y) = z + y;
>implicitdiff(jo, z, y);
```

Tangent Lines and Planes

The tangent plane like the tangent line to a single variable function is based on derivatives, however the partial derivatives are used for the tangent plane. Let's start with the equation of the tangent line to the function $f(x)$ at the point where $x = a$. Recall, the general equation of a line at the point $(a, f(a)$ having slope m is $y - f(a) = m(x - a)$. This can be rewritten knowing that the derivative is the slope of a tangent line as

$$y = f'(a)(x - a) + f(a)$$

. Similarly for a funcion of two variables, the equation of the plane tangent to $z = f(x, y)$ at the point (a, b) has the equation

$$z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$$

. The following examples will show you how the line can easily be translated to Maple syntax.

```
> g := x-> sin(x)-x^3/7+x^2;
> tl := D(g)(5)*(x-5)+g(5);
> plot({g(x),tl},x=-2..8);
```

The above example can help you write a plane equation in Maple.

Critical Points

To find a point where the tangent plane is horizontal, you would need to solve where both first order partials are equal to zero simultaneously.

```
> j := (x, y) -> (x - 2)^3 + 12 + 12*y^2
> solve({diff(j(x,y),x)=0,diff(j(x,y),y)=0},{x,y});
```

Exercises

1. (a) Compute the three distinct second order partial derivatives of

   $$r = 7e^x = 4y^2 + z$$

   using `implicitdiff`

   (b) Compute the three distinct second order partial derivatives of

   $$f(x, y) = \cos(2(y - 1)) + \sin(x - y)$$

   at the point $(\frac{\pi}{6}, \frac{-\pi}{6})$ using the **diff** command and then again using the **D** command.

2. Using the function $f(x, y)$ from Part B above:

a) Plot the function and the plane $x = \frac{\pi}{2}$ on the same graph. Use intervals $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, $-3 \leq z \leq 3$.

b) Graph the two-dimensional intersection of the plane $x = \frac{\pi}{2}$ and $f(x, y)$.

c) Does your two-dimensional graph look like the intersection from your three-dimensional graph? Be sure to use the same ranges to properly compare and rotate the 3-D graph.

d) Find the derivative of $f(x, y)$ in the $x = \frac{\pi}{2}$ plane using first the **diff** command and then the **D** command.

3. For the equation $-(x-2)^2 + 12(4 + y^2) = z + (y + 1)^2$

a) Find the critical point using `implicitdiff`, (along with its z-value) .

b) Plot the equation and the plane along the x-value of the critical point on the same graph. Use intervals $-5 \leq x \leq 10$, $-3 \leq y \leq 3$, $-5 \leq z \leq 150$. Then Graph the two-dimensional intersection. Along this plane, the critical point is what? (maximum, minimum, inflection)

c) Plot the equation and the plane along the y-value of the critical point on the same graph. Then Graph the two-dimensional intersection. Along this plane, the critical point is what?

d) Graph the 3-d equation then state what kind of critical point you found. (maximum, minimum, saddle)