MA143X Theoretical Calculus III Name: Solutions

Final **A Term, 2013**

This is a closed book/notes test. Show all work needed to reach your answers.

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any test/series that you use. If possible, please give the exact value of the series.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+1}$ This is an alternating series +2 with $a_k \searrow 0$, +2 so this series converges +2. But the convergence +2 is conditional (not absolute) since $\sum_{k=1}^{\infty} \frac{1}{k+1}$ diverges by the limit comparison test +2 with the harmonic series.

(b) $\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k = \frac{1}{1-(e/\pi)} = \frac{\pi}{\pi - e}$ +4 because this is a geometric +2 series and $\frac{e}{\pi} < 1$. +2 Since all the terms are positive, this series is absolutely convergent. +2

2. (10 points) For the power series $\sum_{k=0}^{\infty} a_k x^k$, if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ for some $L \in (0, +\infty)$, please describe the interval of convergence for this series. Where is the series uniformly convergent? $a \neq 0$ −2

$\sum_{k=0}^{\infty} a_k x^k$ converges for $x \in \left(-\frac{1}{L}, \frac{1}{L}\right)$ +6 and diverges for $|x| > \frac{1}{L}$ (nothing can be said in general about $|x| = \frac{1}{L}$).

Given any $\epsilon > 0$ +1 $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $\left[-\frac{1}{L}+\epsilon, \frac{1}{L}-\epsilon\right]$ +3.

3. (30 points) Please find the Taylor series with $a = 0$ for each function; you may use the known series for common functions.

(a) $f(x) = \cos(x^3)$

Recall: $\cos(u) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} u^{2k}$ +5

$\forall u \in \mathbb{R}$

+5

Taylor Series: $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k}$

(b) $g(x) = \int_0^x \frac{dt}{1-t^5}$

Recall: $\frac{1}{1-u} = \sum_{k=0}^{\infty} u^k$

when $|u| < 1$ +4

So $g(x) \overset{+2}{=} \int_0^x \sum_{k=0}^{\infty} t^{5k}\, dt$

$\overset{+2}{=} \sum_{k=0}^{\infty} \int_0^x t^{5k}\, dt \overset{+2}{=} \sum_{k=0}^{\infty} \frac{x^{5k+1}}{5k+1}$

provided that $|x| < 1$

Taylor Series: $g(x) = \sum_{k=0}^{\infty} \frac{x^{5k+1}}{5k+1}$

(c) $h(x) = \frac{x^3}{e^x}$ $= x^3 e^{-x}$ +2

Recall: $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$

$\forall u \in \mathbb{R}$ +4

So $h(x) \overset{+2}{=} x^3 \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \overset{+2}{=} \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+3}}{k!}$

$= \sum_{n=3}^{\infty} \frac{(-1)^{n+1} x^n}{(n-3)!}$ either

$n = k+3$

$k = n-3$

Taylor Series: $h(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+3}}{k!}$

4. (8 points) Please give an example of a sequence of functions f_n defined on $[0,1]$ which converges pointwise to $f(x) \equiv 0$, but is *not* uniformly convergent.

Example 1

$$f_n(x) = \begin{cases} x^n & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

Many other examples are also possible.

Example 2

$$f_n(x) = nxe^{-nx^2}$$

(example from class).

5. (12 points) Please complete the proof of the following theorem:

THEOREM: Suppose that $\forall n \in \mathbb{Z}^+$, $a_n \in \mathbb{R}$. Suppose also that $a_n \to L$ for some $L \in \mathbb{R}$. Then the value of L is unique.

PROOF: Suppose that there are two limits, L_1 and L_2. Given (+1) $\epsilon > 0$, since $a_n \to L_1$, $\exists\ N_1 \in \mathbb{Z}^+$ such that (+1) $|a_n - L_1| < \epsilon/2$ $\forall$ (+1) $n > N_1$. In addition, since $a_n \to L_2$, $\exists$ (+1) $N_2 \in \mathbb{Z}^+$ such that (+1) $|a_n - L_2| < \epsilon/2$ $\forall$ (+1) $n > N_2$. Let $N := \max\{N_1, N_2\}$. Then $|L_1 - L_2| \leq |L_1 - a_n| + |a_n - L_2| \overset{(+2)}{<} \epsilon/2 + \epsilon/2 = \epsilon$ $\forall$ (+1) $n > N$. Since $|L_1 - L_2| < \epsilon$ (+1) $\forall\, \epsilon > 0$ (ϵ is arbitary), (+1) $|L_1 - L_2| = 0 \Rightarrow$ (+1) $L_1 = L_2$. QED

6. (10 points) If a sequence $\{a_n\}$ converges to some limit L, please show that every subsequence of this sequence also converges to L.

Since $a_n \to L$, given $\epsilon > 0$, $\exists N \in \mathbb{Z}^+$ such that $|a_n - L| < \epsilon \;\; \forall n > N$. Suppose $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$; then $\forall k \in \mathbb{Z}^+$, $n_k \geq k$. So for $k > N$, also $n_k > N$, implying $|a_{n_k} - L| < \epsilon$

Thus $a_{n_k} \to L$ too. QED

7. (10 points) Please explain why $\displaystyle\int_1^\infty \frac{\ln x}{x^2+x}\,dx$ converges.

Approach 1

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$
$\Leftrightarrow x = e^u$

So

$$\int_1^\infty \frac{\ln x}{x^2+x}\,dx = \int_0^\infty \frac{u}{e^u+1}\,du = \lim_{b\to+\infty}\int_0^b \frac{u\,du}{e^u+1} \leq \lim_{b\to+\infty}\int_0^b \frac{u\,du}{e^u} = \lim_{b\to+\infty}\int_0^b u e^{-u}\,du$$

$$= \lim_{b\to\infty}\left(-ue^{-u}\Big|_0^b + e^{-u}\Big|_0^b\right) = 1 \Rightarrow \int_1^\infty \frac{\ln x}{x^2+x}\,dx \text{ converges.}$$

Approach 2