MA1033 Theoretical Calculus III Name: Solutions

Final A Term, 2017

This is a closed book/notes test. Show all work needed to reach your answers.

A: 85-100
B: 75: 84
C: 62-74
D: 50-61
F: 0-49

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any test/series that you use. If possible, please give the exact value of the series. +2

(a) $\sum_{k=1}^{\infty} \frac{7^k}{k!}$ By the ratio test, +1 since $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left(\frac{7^{k+1}}{(k+1)!} \cdot \frac{k!}{7^k}\right) \overset{+5}{=} \lim_{k\to\infty} \frac{7}{k+1} = 0$, this series is absolutely +2 convergent (all terms are positive).

Since $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\sum_{k=1}^{\infty} \frac{7^k}{k!} = e^7 - 1$ +1

(b) $\sum_{k=1}^{\infty} \left(\frac{1}{-8}\right)^k$ This is a geometric series +2 with $r = -1/8$. +2

Since $|r| < 1$, +1 $\sum_{k=1}^{\infty} \left(\frac{-1}{8}\right)^k \overset{+3}{=} \frac{-1/8}{1+1/8} = -\frac{1}{8+1} = -1/9$ +1

This series is absolutely convergent +1 because $\sum \frac{1}{8}^k$ is also convergent.

2. (10 points) Consider two sequences $\{a_n\}$ and $\{b_n\}$ where $0 < a_n < b_n \,\forall n \in \mathbb{Z}^+$. If the sequence $\{b_n\}$ converges to a finite limit B, what if anything can be said about convergence or divergence for the sequence $\{a_n\}$? Please explain your answer.

Since $a_n < b_n$ and since $b_n \to B < +\infty$, then $a_n \le B$ +3 $\forall n \in \mathbb{Z}^+$. Thus $\{a_n\}$ can +3 not diverge to infinity. But $0 < a_n \le B$ says nothing +4 about whether or not $\{a_n\}$ converges or diverges. For example $\{a_n\}$ might oscillate.

3. (30 points) Please find the Taylor series with $a = 0$ for each function; you may use the known series for common functions. Think carefully about (c) before you write.

(a) $f(x) = \sin(2x)$

+10

Taylor Series: $\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$

(b) $g(x) = \int_0^x t^5 e^t dt$ Since $e^t \overset{+3}{=} \sum_{K=0}^{\infty} \frac{t^K}{K!}$,

$g(x) = \int_0^x \sum_{K=0}^{\infty} \frac{t^{K+5}}{K!} dt = \sum_{K=0}^{\infty} \frac{t^{K+6}}{K!(K+6)} \Big|_0^x$ +3 +3

Taylor Series: $\sum_{K=0}^{\infty} \frac{x^{K+6}}{K!(K+6)}$ +1

(c) $h(x) = \frac{1}{(x-1)(x+1)} = \frac{1}{x^2-1} = \frac{-1}{1-x^2}$ +5

+5

Taylor Series: $-\sum_{K=0}^{\infty} (x^2)^n = -\sum_{K=0}^{\infty} x^{2n}$

4. (8 points) Please decide which of the following statements are true, and which are false (write either "True" or "False" before the statement).

1 point each

(a) False If a sequence $\{a_k\}$ converges to zero, then the series $\sum_{k=1}^{\infty} a_k$ must also converge.

(b) True Geometric series always converge if the absolute value of the constant ratio of consecutive terms is less than one.

(c) True The integral test can be used to show ~~the~~ that certain p-series diverge.

(d) False An improper integral is always defined as a limit, either as the upper limit goes to infinity, or as the lower limit goes to minus infinity.

(e) True If a sequence $\{a_n\}$ satisfies $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1/2$, then the sequence converges to zero.

(f) False If a sequence $\{a_n\}$ with positive terms converges, then the sequence $\{1/a_n\}$ also converges.

(g) True The series $\sum_{k=1}^{\infty} \sin(\frac{\pi}{k})$ diverges.

(h) True $\lim_{n\to\infty} (1 + 1/n)^n = e$.

5. (12 points) Please complete the proof of the following theorem:

THEOREM: Suppose that $\forall n \in \mathbb{Z}^+$, $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}$. Suppose also that $a_n \to A$ for some $A \in \mathbb{R}$, and $b_n \to B$ for some $B \in \mathbb{R}$. Then the sequence $\{a_n b_n\}$ converges to AB.

PROOF: Given +1 $\epsilon > 0$, since $a_n \to A$, $\exists\, N_1 \in \mathbb{Z}^+$ such that +1 $|a_n - A| < \frac{\epsilon}{2(|B|+\epsilon)}$ $\forall$ +1 $n > N_1$. In addition, since $b_n \to B$, $\exists$ +1 $N_2 \in \mathbb{Z}^+$ such that $|b_n - B| < \frac{\epsilon}{2|A|}$ +3 $\forall$ $n > N_2$.

Let $N := \max\{N_1, N_2\}$. Then by the triangle inequality,

$$|a_n b_n - AB| \le |a_n b_n - A b_n| + |A b_n - AB| \le |b_n||a_n - A| + \text{+1 } |A|\,|b_n - B|.$$

Since $|b_n| \le |B| + \epsilon$, +4 $|a_n b_n - AB| \le (|B|+\epsilon)\frac{\epsilon}{2(|B|+\epsilon)} + |A|\frac{\epsilon}{2|A|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall n > N$. Thus $a_n b_n \to AB$. QED

6. (10 points) If a sequence $\{a_n\}$ converges to some limit L, please prove that every subsequence of this sequence also converges to L.

+4

Since $a_n \to L$, given $\epsilon > 0$, $\exists\, N \in \mathbb{Z}^+$ such that $|a_n - L| < \epsilon \ \forall\, n > N$. Consider a subsequence $\{a_{n_k}\}$ of $\{a_n\}$. Because this is a subsequence, $n_k \geq K \ \forall\, K \in \mathbb{Z}^+$. +3

Hence $|a_{n_k} - L| < \epsilon$ +3 $\forall\, n_k \geq K > N$, and thus $a_{n_k} \to L$.

(QED)

7. (10 points) For the power series $\sum_{k=0}^{\infty} a_k x^k$, if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ for some $L \in (0, +\infty)$, please find the interval of convergence for the series $\sum_{k=0}^{\infty} (a_k)^2 x^k$. Show all work needed to reach your answer.

+2 +2 +1

Notice that $\lim_{K\to\infty} \frac{(a_{K+1})^2}{(a_K)^2} = \left(\lim_{K\to\infty} \frac{a_{K+1}}{a_K}\right)\left(\lim_{K\to\infty} \frac{a_{K+1}}{a_K}\right) = L \cdot L = L^2.$

+5

Thus the power series $\sum_{K=0}^{\infty} (a_K)^2 x^K$ converges on $\left(-\frac{1}{L^2}, \frac{1}{L^2}\right)$.