MA1033 Theoretical Calculus III Name Solutions

Quiz 2 A Term, 2017

High: 20
Median: 18
Low: 10

Show all work needed to reach your answers.

(5 points each) As $n \to \infty$, please find the value of the limit L if the sequence converges, or find that the sequence diverges. Also name any test or rule that you use.

1. $a_n = \dfrac{n!}{n^n}$

$= \frac{1}{n}\left[\frac{2}{n}\cdot\frac{3}{n}\cdot\ldots\cdot\frac{(n-1)}{n}\cdot\frac{n}{n}\right]$ (+1)

Since $0 \le a_n \le \frac{1}{n}$ $\forall n \in \mathbb{Z}^+$ (+2)

by the squeeze thm, (+1)

$a_n \to 0$ (+1)

By the ratio test, (+1)

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n^n}{(n+1)^n}\right)$ (+1)

$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^n = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$ (+1) Shown in class.

$\Rightarrow a_n \to 0$ (+1)

0

2. $a_n = n\sin(2/n)$

$= 2\left(\frac{\sin(2/n)}{2/n}\right)$ (+2)

Since we showed in class that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ (+2), this limit must be

$\lim_{n\to\infty} a_n = 2\lim_{n\to\infty}\frac{\sin(2/n)}{2/n} = 2\cdot 1$

by letting $\theta = 2/n$ (+1)

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3. $a_n = \dfrac{\cos n\pi}{n+1} = \begin{cases} \frac{1}{n+1} & n \text{ is even} \\ \frac{-1}{n+1} & n \text{ is odd} \end{cases}$ (+2)

So by the squeeze thm, (+1)

$0 \le |a_n| = \frac{1}{n+1} \to 0$ (+1)

$\Rightarrow a_n \to 0$ (+1)

0

4. $a_n = \left(1+\frac{1}{n^2}\right)^n$

Consider $\ln(a_n) = n\ln\left(1+\frac{1}{n^2}\right) = \frac{\ln(1+\frac{1}{n^2})}{1/n}$ (+1); it's of the form "$\frac{0}{0}$". (+1)

Using l'Hopital, $\frac{\frac{1}{1+\frac{1}{n^2}}\left(\frac{-2}{n^3}\right)}{-1/n^2}$ (+1) $= \frac{2}{n+1/n}$ has the same limit.

Since $\frac{2}{n+1/n} \to 0$ as $n\to\infty$, (+1)

$\lim_{n\to\infty}\ln(a_n) = 0$ too $\Rightarrow \lim_{n\to\infty} a_n = 1$. (+1)

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