MA1033 Theoretical Calculus III Name Solutions

Quiz 6 A Term, 2015

(5 points each) Show all work needed to reach your answers. If you need the value of any limit to reach your answer, please compute that value. For questions #1 and #2, please find a power series representation for each function, then for #1, give the radius of convergence.

1. $f(x) = \dfrac{x}{1-x}$

Since $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| \le 1$ +2 +1

High: 19
Median: 15
Low: 6

Series: $\sum_{k=0}^{\infty} x^{k+1}$ +2 $R =$ 1

2. $f(x) = \int_0^x \frac{\sin t}{t} dt$

Since $\sin t = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} t^{2k-1}$ +1, $\frac{\sin t}{t} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} t^{2k-2}$ +1,

then $\int_0^x \frac{\sin t}{t} dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} \int_0^x t^{2k-2} dt$ +1

$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} \left(\frac{t^{2k-1}}{2k-1} \right|_0^x$ +1

Series: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!\,(2k-1)}$ +1

3. Consider the power series $\sum_{k=0}^{\infty} a_k (x-a)^k$. If this power series equals $f(x)$ for some $f \in C^{\infty}$, what are the values of the coefficients a_k in terms of f and its derivatives? Based on this power series, what power series represents $f'(x)$.

+2 $a_k = \dfrac{f^{(k)}(a)}{k!}$

+3 either

$f'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1} = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{(k-1)!} (x-a)^{k-1}$

4. If one wishes to approximate e^{-x} on the interval $[0,1]$ by its Taylor polynomial $P_n(x, x_0)$ with center $x_0 = 0$, what is a bound on the error ?

$e^{-x} - P_n(x,0) \overset{+2}{=} R_n(x,0)$ Taylor Remainder

Error +1 $\le |R_n(x,0)|$ +1 $= \left| \frac{f^{(n+1)}(c)\, x^n}{(n+1)!} \right|$ +1 $\le \frac{1}{(n+1)!}$

for $x \in [0,1]$

One can also notice that this is an alternating series.