MA1033 Theoretical Calculus III Name: ______________________

Final **A Term, 2017**

This is a closed book/notes test. Show all work needed to reach your answers.

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any test/series that you use. If possible, please give the exact value of the series.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{7^k}{k!}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{-8}\right)^k$

2. (10 points) Consider two sequences $\{a_n\}$ and $\{b_n\}$ where $0 < a_n < b_n \, \forall \, n \in \mathbb{Z}^+$. If the sequence $\{b_n\}$ converges to a finite limit B, what if anything can be said about convergence or divergence for the sequence $\{a_n\}$? Please explain your answer.

3. (30 points) Please find the Taylor series with $a = 0$ for each function; you may use the known series for common functions. Think carefully about (c) before you write.

   (a) $f(x) = \sin(2x)$

   Taylor Series: ____________________

   (b) $g(x) = \int_0^x t^5 e^t \, dt$

   Taylor Series: ____________________

   (c) $h(x) = \dfrac{1}{(x-1)(x+1)}$

   Taylor Series: ____________________

4. (8 points) Please decide which of the following statements are true, and which are false (write either "True" or "False" before the statement).

   (a) ________ If a sequence $\{a_k\}$ converges to zero, then the series $\sum_{k=1}^{\infty} a_k$ must also converge.

   (b) ________ Geometric series always converge if the absolute value of the constant ratio of consecutive terms is less than one.

   (c) ________ The integral test can be used to show the certain p-series diverge.

   (d) ________ An improper integral is always defined as a limit, either as the upper limit goes to infinity, or as the lower limit goes to minus infinity.

   (e) ________ If a sequence $\{a_n\}$ satisfies $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1/2$, then the sequence converges to zero.

   (f) ________ If a sequence $\{a_n\}$ with positive terms converges, then the sequence $\{1/a_n\}$ also converges.

   (g) ________ The series $\sum_{k=1}^{\infty} \sin(\frac{\pi}{k})$ diverges.

   (h) ________ $\lim_{n\to\infty} (1 + 1/n)^n = e$.

5. (12 points) Please complete the proof of the following theorem:

   THEOREM: Suppose that $\forall\, n \in \mathbb{Z}^+$, $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}$. Suppose also that $a_n \to A$ for some $A \in \mathbb{R}$, and $b_n \to B$ for some $B \in \mathbb{R}$. Then the sequence $\{a_n b_n\}$ converges to AB.

PROOF: Given ____________, since $a_n \to A$, $\exists\ N_1 \in \mathbb{Z}^+$ such that ____________ $< \frac{\epsilon}{2(|B|+\epsilon)}$ $\forall$ ____________. In addition, since $b_n \to B$, $\exists$ ____________ such that ____________ $\forall$ ____________.

Let $N := \max\{N_1, N_2\}$. Then by the triangle inequality,

$|a_n b_n - AB| \leq |a_n b_n - A b_n| + |A b_n - AB| \leq |b_n||a_n - A| +$ ____________.

Since $|b_n| \leq |B| + \epsilon$, ________________________

____. QED

6. (10 points) If a sequence $\{a_n\}$ converges to some limit L, please prove that every subsequence of this sequence also converges to L.

7. (10 points) For the power series $\sum_{k=0}^{\infty} a_k x^k$, if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ for some $L \in (0, +\infty)$, please find the interval of convergence for the series $\sum_{k=0}^{\infty} (a_k)^2 x^k$. Show all work needed to reach your answer.