MA1033 Theoretical Calculus III **Name:** ______________________________

Final **A Term, 2018**

This is a closed book/notes test. Show all work needed to reach your answers.

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any test/series that you use. If possible, please give the exact value of the series.

   (a) $\displaystyle\int_0^1 \frac{dx}{x^2}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1}$

2. (10 points) For the power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$, if $\displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ for some $L \in (0, +\infty)$, please describe the interval of convergence for this series. Where is the series uniformly convergent?

3. (30 points) Please find the Taylor series with $x_0 = 0$ for each function; you may use the known series for common functions.

   (a) $f(x) = \ln(1 + x^2)$

   Taylor Series: ______________________

   (b) $g(x) = \dfrac{d}{dx}\left(\dfrac{\sin x^2}{x}\right)$

   Taylor Series: ______________________

   (c) $h(x) = \displaystyle\int_0^x \frac{t}{e^t}\,dt$

   Taylor Series: ______________________

4. (10 points) Suppose that $f_n(x) := \dfrac{nx}{n^2x^2+4}$. Notice that f_n converges pointwise to zero for $x \in [0, +\infty)$; please explain why it does not converge uniformly.

5. (10 points) Please complete the proof of the following theorem:

THEOREM: Suppose that $\{a_n\}$ is a sequence of real numbers, and suppose that there is some $L \in \mathbb{R}$ such that any subsequence $\{a_{n_j}\}$, the subsequence converges to L. Prove that $\{a_n\}$ converges to L.

PROOF (Contradiction): Suppose that the sequence does not converge.

Then for some $\epsilon > 0$, for any $N_1 \in \mathbb{Z}^+$, there is an n_1 ____ N_1 such that $|a_{n_1} - L|$ ________. Now pick $N_2 = n_1 + 1$. Again since $\{a_n\}$ does not converge, ________________ such that $|a_{n_2} - L|$ ________. Next pick $N_3 =$ ________. Since $\{a_n\}$ does not converge, ____________________ ________________________. In this way, we can construct a subsequence which ____________________________. This contradicts that ____.

QED

6. (10 points) If a sequence $\{a_n\}$ converges to some limit L, please show that this limit is unique (*i.e.* that there is only one value for L).

7. (10 points) Does $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{x^5 + x}\, dx$ converge or diverge? Please explain why.