MA1033 Theoretical Calculus III Name: ______________________________

Final **A Term, 2019**

This is a closed book/notes test. Show all work needed to reach your answers.

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any tests that you use. If possible, please name the series and give its exact value.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$

   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^5}$

2. (5 points) Please state the least upper bound axiom (or least upper bound property) for the real numbers.

3. (30 points) Please find the Taylor series with $x_o = 0$ for each function; you may use the known series for common functions.

   (a) $f(x) = \cos(\sqrt{x})$

   Taylor Series: ______________________

   (b) $g(x) = \dfrac{x}{1+x}$

   Taylor Series: ______________________

   (c) $h(x) = \dfrac{d}{dx}\left(\dfrac{e^{x^2} - 1 - x^2}{x^4}\right)$

   Taylor Series: ______________________

4. (10 points) Please evaluate the following integral, or explain why the integral does not converge:

$$\int_0^5 \frac{dx}{(3-x)^2}$$

5. (15 points) Please complete the proof of the following theorem:

THEOREM: Suppose that $\forall\, n \in \mathbb{Z}^+$, $a_n, b_n, c_n \in \mathbb{R}$ and

$$a_n < b_n < c_n \ .$$

Suppose also that $a_n \to L$ and $c_n \to L$ as $n \to \infty$ for some $L \in \mathbb{R}$. Then the sequence $\{b_n\}$ converges to L.

PROOF: Given ____________, since $a_n \to L$, $\exists\ N_a \in \mathbb{Z}^+$ such that ____________ $< \epsilon \quad \forall$ ____________ . Also since $c_n \to L$, $\exists$ ____________ such that ________________ $\forall$ ____________ . Let $N := \max\{N_a, N_c\}$. Then $\forall\ n > N$, ______________ $< c_n - L \leq |c_n - L| < \epsilon$. In addition,

$L - b_n <$ ____ .

Therefore $|b_n - L|$ ____________, implying that $b_n \to L$. QED

6. (10 points) Suppose $\sum_{k=0}^{\infty} a_k(x-x_o)^k$ converges with radius of convergence $R_1 = 3$.

What's the radius of convergence R for the series $\sum_{k=0}^{\infty} \frac{2a_k}{5^k}(x-x_o)^k$?

7. (10 points) Consider the alternating series $\sum_{k=1}^{\infty}(-1)^k a_k$ where, for $k \in \mathbb{Z}^+$,

$$a_k = \begin{cases} \frac{2}{k+1} & k \text{ is odd} \\ \frac{4}{k^2} & k \text{ is even} \end{cases}$$

Does this series converge absolutely, converge conditionally, or diverge? Please carefully explain your answer. Hint: Consider the partial sums.