MA1033 Theoretical Calculus III Name: Solutions

Final **A Term, 2018**

This is a closed book/notes test. Show all work needed to reach your answers.

1. (20 points) For each of the following series, please determine whether the series converges absolutely, converges conditionally or diverges, and please name any test/series that you use. If possible, please give the exact value of the series.

   (a) $\int_0^1 \frac{dx}{x^2}$ $= \lim_{b\to 0} \int_b^1 \frac{dx}{x^2}$ (+2) $= \lim_{b \to 0} \left(\frac{x^{-1}}{-1} \Big|_b^1 \right)$ (+2) $= \lim_{b\to 0}\left(\frac{1}{b} - 1\right)$ (+2) $= +\infty$ (+2)

   So this integral diverges to infinity. (+2)

A: 82-100
B: 70-81
C: 56-69
D: 41-55
F: 0-40

   (b) $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k+1}$

   - Decreasing (+1)
   - a_K goes to zero (+1)

   Since $a_K = \frac{1}{K+1} \searrow 0$ and since this series (+1) alternates, by the alternating series (+1) test, this series converges.

   Since $\sum_{K=1}^{\infty} \left| \frac{(-1)^{K-1}}{K+1} \right| = \sum_{K=1}^{\infty} \frac{1}{K+1}$ (+1) and since (+2) $\frac{1}{2K} \leq \frac{1}{K+1}$ (+1), convergence is conditional (+1) by direct comparison with the harmonic series. (+1)

   Also $\sum_{n=2}^{\infty} \frac{(-1)^n}{n} = -\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - 1 \right) = -(\ln(2) - 1) = 1 - \ln(2)$ (+1) Bonus Point

   $n = K+1$

2. (10 points) For the power series $\sum_{k=0}^{\infty} a_k x^k$, if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ for some $L \in (0, +\infty)$, please describe the interval of convergence for this series. Where is the series uniformly convergent? $\boxed{X_0 \neq 0 \ (-2)}$ (+6)

   $\sum_{K=0}^{\infty} a_K$ converges absolutely for $x \in \left(-\frac{1}{L}, \frac{1}{L}\right)$ and diverges for $|x| > \frac{1}{L}$. Given $\epsilon > 0$ (+1), this series converges uniformly on $\left[-\frac{1}{L} + \epsilon, +\frac{1}{L} - \epsilon\right]$

   (+3)

3. (30 points) Please find the Taylor series with $\cancel{a}\, x_0 = 0$ for each function; you may use the known series for common functions.

(a) $f(x) = \ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x^2)^n$ +6

+3

Taylor Series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n}$ +1

+1

(b) $g(x) = \frac{d}{dx}\left(\frac{\sin x^2}{x}\right) = \frac{d}{dx}\left(\frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1}\right)$ +4

$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+1}\right)$ +2

$= \sum_{n=0}^{\infty} (-1)^n \frac{(4n+1)}{(2n+1)!} x^{4n}$

+2

Taylor Series: $\sum_{n=0}^{\infty} (-1)^n \frac{4n+1}{(2n+1)!} x^{4n}$ +1

+1

(c) $h(x) = \int_0^x \frac{t}{e^t}\, dt = \int_0^x t e^{-t}\, dt = \int_0^x t \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}\, dt$ +4

+1

$= \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n!}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)n!} t^{n+2} \Big|_0^x$

+2

+1 evaluate

Taylor Series: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)n!} x^{n+2}$ +1

4. (10 points) Suppose that $f_n(x) := \dfrac{nx}{n^2x^2+4}$. Notice that f_n converges pointwise to zero; please explain why it does not converge uniformly.

for $x \in [0, +\infty)$

Notice that $f_n'(x) = \dfrac{n(n^2x^2+4) - nx(2n^2x)}{(n^2x^2+4)^2} = 0$ (+1) set (+1)

$\iff n^3x^2 + 4n - 2n^3x^2 = 0 \iff 4 = n^2x^2 \iff 2 = nx$ (+2) (+2)

So $x = 2/n$ is a critical point, and $f_n\left(\frac{2}{n}\right) = \frac{2}{4+4} = \frac{1}{4}$ (+1) (+1)

Hence $f_n \not\xrightarrow{u} 0$ since each f_n reaches $\frac{1}{4}$ at $x = \frac{2}{n}$. (+2)

5. (10 points) Please complete the proof of the following theorem:

THEOREM: Suppose that $\{a_n\}$ is a sequence of real numbers, and suppose that there is some $L \in \mathbb{R}$ such that any subsequence $\{a_{n_j}\}$, the subsequence converges to L. Prove that $\{a_n\}$ converges to L.

PROOF (Contradiction): Suppose that the sequence does not converge.

Then for some $\epsilon > 0$, for any $N_1 \in \mathbb{Z}^+$, there is an n_1 $>$ (+1) N_1 such that $|a_{n_1} - L|$ $\geq \epsilon$ (+1). Now pick $N_2 = n_1 + 1$. Again since $\{a_n\}$ does not converge, $\exists\, n_2 > N_2$ (+1) such that $|a_{n_2} - L|$ $\geq \epsilon$ (+1). Next pick $N_3 =$ $n_2 + 1$ (+1). Since $\{a_n\}$ does not converge, $\exists\, n_3 > N_3$ such that $|a_{n_3} - L| \geq \epsilon$ (+2). In this way, we can construct a subsequence which does not converge to L (+4). This contradicts that every subsequence converges to L. (+2)

QED

6. (10 points) If a sequence $\{a_n\}$ converges to some limit L, please show that this limit is unique (*i.e.* that there is only one value for L).

+1

Suppose the limit is not unique.

Let $a_n \to L_1$ +1 and $a_n \to L_2$. Then given any +1 $\epsilon > 0$,
$\exists N_1 \in \mathbb{Z}^+$ such that +1 $|a_n - L_1| < \epsilon/2 \;\; \forall n > N_1$, and
$\exists N_2 \in \mathbb{Z}^+$ such that +1 $|a_n - L_2| < \epsilon/2 \;\; \forall n > N_2$.
Define $N := $ +1 $\max\{N_1, N_2\}$. Then if $n > N$

+1 +1

$$|L_1 - L_2| \le |L_1 - a_n| + |a_n - L_2| < \epsilon/2 + \epsilon/2 = \epsilon.$$

+1

Since this works $\forall \epsilon > 0$, it must be that

$L_1 = L_2$. +1

7. (10 points) Does $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{x^5 + x}\, dx$ converge or diverge? Please explain why.

+4

By the integral test, this integral converges iff the series $\displaystyle\sum_{K=3}^{\infty} \frac{\sin K}{K^5 + K}$ converges. Since $\displaystyle\frac{|\sin K|}{K^5 + K} \le$ +2 $\displaystyle\frac{1}{K^5 + K} < \frac{1}{K^5}$,
this series converges +4 by direct comparison to $\displaystyle\sum_{K=3}^{\infty} \frac{1}{K^5}$ (a p-series with $p > 1$).

By definition, $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{x^5 + x}\, dx = \lim_{b \to \infty} \int_{\pi}^{b} \frac{\sin x}{x^5 + x}\, dx$. Now $\displaystyle\int_{\pi}^{b} \left|\frac{\sin x}{x^5 + x}\right| dx \le \int_{\pi}^{b} \frac{dx}{x^5} = \frac{1}{4}\left(\frac{1}{\pi^4} - \frac{1}{b^4}\right)$

As $b \to 0$, this is an increasing sequence (function) bounded above (by $\frac{1}{4\pi^4}$) hence it must converge. So this integral is absolutely convergent, and hence convergent.