MA1033 Theoretical Calculus III Name Solutions

Quiz 2 A Term, 2015

High: 20
Median: 19
Low: 13

Show all work needed to reach your answers.

(5 points each) As $n \to \infty$, please find the value of the limit L if the sequence converges, or find that the sequence diverges. Also name any test or rule that you use.

1. $a_n = \dfrac{n^2}{n!}$

By the Ratio Test, +1

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!}\frac{n!}{n^2} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0 < 1,$ +2 +1

$\lim_{n\to\infty} a_n = 0$ +1

2. $a_n = (-1)^n \dfrac{3n}{n+3}$

Here $\underbrace{a_{2n}}_{\text{even elements}} \to 3$, while $\underbrace{a_{2n+1}}_{\text{odd elements}} \to -3$ +4

So this sequence oscillates.

$\lim_{n\to\infty} (-1)^n \frac{3n}{n+3}$ Diverges +1

3. $a_n = \left(\dfrac{1}{n}\right)^{1/n}$

Let $b_n = \ln(a_n)$. Then using l'Hôpital's rule, +1 one sees that

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{1}{n}\right) = -\lim_{n\to\infty} \frac{\ln n}{n} = -\lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ +1 +1 +1

Finally, if $b_n \to 0$, then $a_n \to 1$ +1

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1$

4. $a_n = \dfrac{(n^2)!}{(n!)^2}$

Because of the powers and factorials, try Ratio Test: +1

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2!\,(n!)^2}{((n+1)!)^2\, n^2!} = \lim_{n\to\infty} \frac{(n+1)^2!\ \cancel{n^2\cdot(n-1)^2\cdot\ldots\cdot 2^2\cdot 1^2}}{n^2!\ (n+1)^2\cdot \cancel{n^2\cdot(n-1)^2\cdot\ldots\cdot 2^2\cdot 1^2}}$ +1 +1

$= \lim_{n\to\infty} \frac{\cancel{(n+1)^2}\cdot[(n+1)^2-1]\cdot[(n+1)^2-2]\cdot\ldots\cdot 2\cdot 1}{n^2\cdot[n^2-1]\cdot[n^2-2]\cdot\ldots\cdot 2\cdot 1\ \cancel{(n+1)^2}}$

$= \lim_{n\to\infty} [(n+1)^2-1][(n+1)^2-2]\cdot\ldots\cdot(n^2+1)$ +1

$= +\infty$

$\lim_{n\to\infty} \frac{(n^2)!}{(n!)^2}$ +1 Diverges by the Ratio Test.