MA144X Theoretical Calculus IV   Name: Solutions

Final   **B Term, 2013**

Show all work needed to reach your answers.

1. (10 points) For $g(x, y, z) = 3xy^2 \cos yz + x/y$, please compute $\partial g/\partial y$

Form: +1

+3 +3 +3

$\partial g/\partial y =$ $6xy \cos yz - 3xy^2 z \sin yz - \frac{x}{y^2}$

2. (10 points) For the surface $z = f(x, y) = x^2 - 3xy + 7x - 3y^2 + 8$, please find the critical point and decide if it is a maximum, minimum or a saddle point.

$\partial_x f(x,y) \overset{+1}{=} 2x - 3y + 7 = 0$

set +1

$\partial_y f(x,y) \overset{+1}{=} -3x - 6y = 0 \Rightarrow x + 2y = 0 \Rightarrow x = -2y$ +1

Plug back in: $-4y - 3y + 7 = 0$ +1 $\Rightarrow y = 1$ +1

$D(f) = (\partial_x^2 f)(\partial_y^2 f) - (\partial_{xy} f)^2 < 0$ +2

Critical Point: $(-2, 1)$ +1

Type: Saddle Point +1

3. (12 points) Please find an equation of the plane passing through the points $(1, 2, 0)$, $(0, 4, 1)$ and $(8, 0, 1)$.

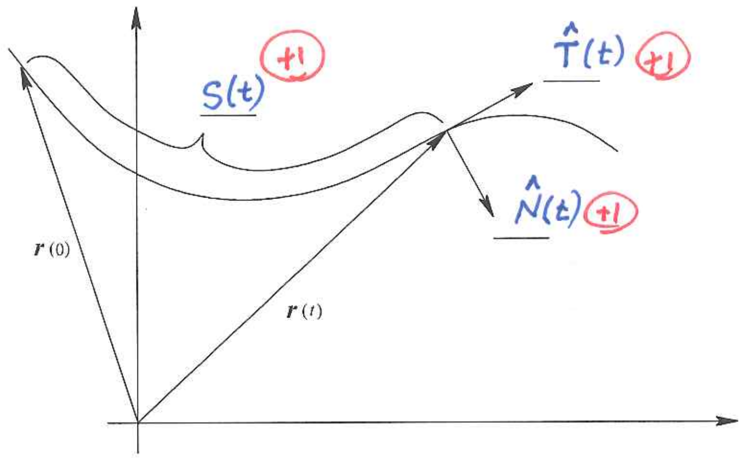
Find $\vec{N}$: $\vec{x}_1 = \langle 1-0, 2-4, 0-1 \rangle \overset{+2}{=} \langle 1, -2, -1 \rangle \Rightarrow \langle -1, 2, 1 \rangle$

$\vec{x}_2 = \langle 8-0, 0-4, 1-1 \rangle \overset{+2}{=} \langle 8, -4, 0 \rangle \Rightarrow \langle 2, -1, 0 \rangle$

$\vec{N} \overset{+2}{=} \vec{x}_1 \times \vec{x}_2 = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ 2 & -1 & 0 \end{vmatrix} \overset{+2}{=} \langle +1, 2, -3 \rangle \Rightarrow \pi: 1(x-1) + 2(y-2) - 3z = 0$ +4

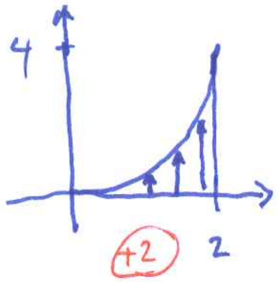
Plane: $x + 2y - 3z = +5$

4. (18 points) Please complete the following table and diagram:



Symbol	Defined as (in symbols)	Name
$\mathbf{T}(t)$	+2 $\frac{\vec{v}(t)}{\lvert\vec{v}(t)\rvert} = \frac{\vec{v}(t)}{s'(t)}$	+1 unit tangent vector
$\mathbf{N}(t)$	$\frac{d\mathbf{T}/ds}{\lvert d\mathbf{T}/ds\rvert}$	+1 unit normal vector
$s'(t)$	+2 $\lvert\vec{v}(t)\rvert$	+2 speed
+1 $K(s)$	+2	curvature
$\mathbf{v}(t)$	+2 $\vec{r}\,'(t)$	+2 velocity vector

5. (15 points) Please compute $\iint_D \frac{\sqrt{1+x^2}}{x}\,dA$ over the region D between the curves $y = x^2$, $y = 0$ and $x = 2$.



$$\iint_D \frac{\sqrt{1+x^2}}{x}\,dA = \int_0^2 \int_0^{x^2} \frac{\sqrt{1+x^2}}{x}\,dy\,dx = \int_0^2 \frac{\sqrt{1+x^2}}{x} \int_0^{x^2} dy\,dx$$

+6 +1

$$= \int_0^2 \frac{\sqrt{1+x^2}}{\cancel{x}}\,x^{\cancel{2}}\,dx = \int_0^2 \sqrt{1+x^2}\,x\,dx = \frac{1}{2}\int_1^5 \sqrt{u}\,du$$

+2 +1

$u = 1+x^2$ +1

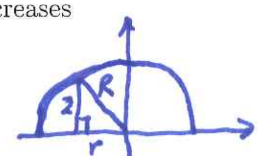
$du = 2x\,dx$

$$= \frac{2}{3}\,\frac{1}{2}\,u^{3/2}\Big|_1^5 = \frac{1}{3}\left(5^{3/2}-1\right)$$

+1 +1

$\frac{1}{3}\left(5^{3/2}-1\right)$

6. (15 points) Consider a hemispherical dome with radius R sitting on top of the x,y-plane: $x^2+y^2+z^2 = R^2$, $z \geq 0$. Suppose that the dome is filled with a gas whose density decreases linearly with height (so $\delta(z) = \delta_0(1 - z/R)$). Please find the mass of this dome.



$$\text{Mass} = \iiint_\Omega \delta(z)\,dV = \int_0^{2\pi}\int_0^R \int_0^{\sqrt{R^2-r^2}} \delta_0\left(1-\frac{z}{R}\right)dz\,r\,dr\,d\theta$$

+2 +6

Cylindrical

$$= 2\pi\delta_0 \int_0^R \left(z - \frac{z^2}{2R}\right)\Bigg|_0^{\sqrt{R^2-r^2}} r\,dr = 2\pi\delta_0\int_0^R \left(\sqrt{R^2-r^2} - \frac{(R^2-r^2)}{2R}\right) r\,dr$$

$z^2 = R^2 - r^2$

$z = \sqrt{R^2-r^2}$

+1 +1 +1

$$= \cancel{2}\pi\delta_0 \left[-\frac{2}{3}(R^2-r^2)^{3/2}\Big|_0^R + \frac{(R^2-r^2)^2}{4R}\Big|_0^R\right] = \pi\delta_0\left[\frac{2}{3}R^3 - \frac{R^3}{4}\right] = \frac{5\pi\delta_0}{12}R^3$$

+1 +1 +1 +1

either

$$\text{Mass} = \iiint_\Omega \delta(z)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R \delta_0\left(1 - \frac{\rho\cos\varphi}{R}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

+2 +6

Spherical

$$= 2\pi\delta_0\left[\int_0^{\frac{\pi}{2}} \sin\varphi\,d\varphi \int_0^R \rho^2\,d\rho - \frac{1}{R}\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\,d\varphi \int_0^R \rho^3\,d\rho\right]$$

+1 +1 +1

$$= 2\pi\delta_0\left[-\cos\varphi\Big|_0^{\pi/2}\,\frac{\rho^3}{3}\Big|_0^R - \frac{1}{R}\,\frac{\sin^2\varphi}{2}\Big|_0^{\pi/2}\,\frac{\rho^4}{4}\Big|_0^R\right] = 2\pi\delta_0\left[\frac{R^3}{3} - \frac{R^3}{8}\right] = \frac{5\pi\delta_0 R^3}{12}$$

+1 +1 +1 +1

Mass: $\frac{5\pi\delta_0 R^3}{12}$

7. (10 points) If a function $f : [a, b] \to \mathbb{R}$ is *continuous* at a point $x_0 \in [a, b]$, what is the δ-ϵ definition of *continuous*? Hint: Start with "Given $\epsilon > 0$, . . ."

+1 +1 +1 +1 +2 +1

Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. if $d(x, x_0) < \delta$, then

+3

$d(f(x), f(x_0)) < \epsilon$.

8. (10 points) Suppose $f(x, y) = 0$ is a smooth curve in $\mathbb{R}^2$ (the x, y-plane), so that f is a differentiable function. Please explain why $\nabla f(x, y)$ is perpendicular to the curve at (x, y).

+2

Suppose the curve is parameterized by $(x(t), y(t))$ for $t \in \mathbb{R}$. Then $f(x(t), y(t)) \overset{+1}{=} 0\ \forall t \in \mathbb{R}$.

So $0 \overset{+1}{=} \frac{d}{dt}\left(f(x(t), y(t))\right) \overset{+2}{=} \vec{\nabla} f(x(t), y(t)) \cdot \overset{+2}{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}$

Chain Rule

Gradient ∇f 1
Dot Product 3
$\nabla f \cdot \_ = 0$ 4
$\nabla f \cdot \vec{v} = 0$ 7

Thus $\vec{\nabla} f(x(t), y(t)) \cdot \vec{v}(t) = 0$, and since $\vec{v}(t)$ is tangent +1 to the curve for all t, $\vec{\nabla} f$ must be $\perp$ +1 to the curve.