MA1034 Theoretical Calculus IV Name: Solutions

Final **B Term, 2017**

Show all work needed to reach your answers.

1. (10 points) For the surface $z = f(x,y) = 3x^2 + xy - 7y - 3y^2 + 2$, please find an equation for the tangent plane at $(1,-1,8)$.

Exam
A: 86 - 100 (4)
B: 73 - 85 (4)
C: 60 - 72 (3)
D: 47 - 59 (2)
F: 0 - 46 (0)

Tangent Plane: $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ (+2)

$f(1,-1) = 8$ (+1)

$f_x(1,-1) = 6x + y\big|_{(1,-1)} = 5$ (+1)

$f_y(1,-1) = x - 7 - 6y\big|_{(1,-1)} = 0$ (+1)

High: 97
Median: 76
Low: 47

Tangent Plane: $z = 8 + 5(x-1) + 0(y+1) \iff z = 5x + 3$ (+1) (+1) (+2) (+1)

2. (10 points) Consider the vector-valued function $\boldsymbol{f}(x,t) = \langle x\cos(\omega t), x\sin(\omega t)\rangle$ where ω is a constant. Please the Jacobian matrix $J(\boldsymbol{f})(x,t)$

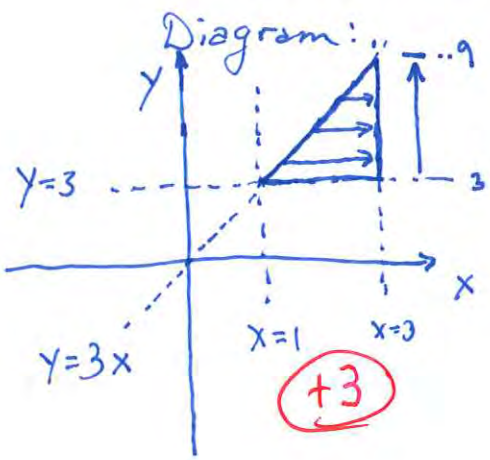
$$J(f)(x,t) = \begin{bmatrix} (\partial_x f_1)(x,t) & (\partial_t f_1)(x,t) \\ (\partial_x f_2)(x,t) & (\partial_t f_2)(x,t) \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\omega x \sin(\omega t) \\ \sin(\omega t) & \omega x \cos(\omega t) \end{bmatrix}$$

$$J(\boldsymbol{f})(x,t) = \begin{bmatrix} \cos(\omega t) & -\omega x \sin(\omega t) \\ \sin(\omega t) & \omega x \cos(\omega t) \end{bmatrix}$$

(+2) (+3) (+2) (+3)

3. (10 points) Please reverse the order of integration for $\displaystyle\int_1^3 \int_3^{3x} f(x,y)\,dy\,dx$

Diagram:



$$\int_3^9 \int_{y/3}^3 f(x,y)\,dx\,dy$$

(+1) (+2) (+1) (+2) (+1)

4. (10 points) For the vector function $x(t) = \langle 5t, t^2+t, e^{-3t}\rangle$ please find the unit tangent vector $T(t)$.

$$\hat{T}(t) = \frac{\vec{x}'(t)}{|\vec{x}'(t)|} = \frac{\langle 5, 2t+1, -3e^{-3t}\rangle}{\sqrt{25+(2t+1)^2+9e^{-6t}}}$$

+6 +4

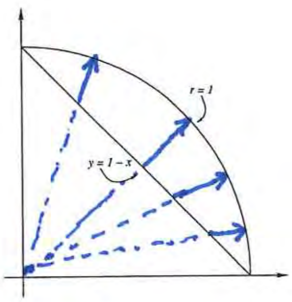
$$\frac{\langle 5, 2t+1, -3e^{-3t}\rangle}{\sqrt{26+4t+4t^2+9e^{-6t}}}$$

5. (20 points)

(a) Please set up the following double integral as an iterated integral in polar coordinates:

$$\iint_D f(r,\theta)\,dA$$

where D is the region shown in the figure below:



$y = 1 - x$

$r\sin\theta = 1 - r\cos\theta$

$\iff r(\sin\theta + \cos\theta) = 1$

$$\int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} f(r,\theta)\,\underbrace{r\,dr\,d\theta}_{+1}$$

+2 +2 +3 +2

(b) For $f(r,\theta) = (\sin\theta+\cos\theta)^2$, please evaluate this integral.

$$\int_0^{\pi/2}\int_{\frac{1}{\cos\theta+\sin\theta}}^{1} (\sin\theta+\cos\theta)^2\, r\,dr\,d\theta = \int_0^{\pi/2} (\sin\theta+\cos\theta)^2 \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r\,dr\,d\theta$$

+2

$$= \frac{1}{2}\int_0^{\pi/2} (\sin\theta+\cos\theta)^2 \left[1^2 - \left(\frac{1}{\cos\theta+\sin\theta}\right)^2\right] d\theta = \frac{1}{2}\int_0^{\pi/2} (\sin\theta+\cos\theta)^2 - 1\,d\theta$$

+2 +2

$$= \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} = 1/2$$

+2 +1 +1

1/2

6. (10 points) For the surface $z = f(x, y) = x^2 + xy + y^2 + x - y - 10$ please find and classify the critical point.

Since f is a polynomial, the critical point must be where (+1) $\vec{\nabla} f(x_0, y_0) = 0$:

set

$$\vec{\nabla} f(x,y) = \langle 2x + y + 1,\ x + 2y - 1 \rangle \overset{(+1)}{=} \vec{0}$$

$$\Leftrightarrow \begin{cases} 2x+y+1=0 \\ x+2y-1=0 \end{cases} (+1) \Leftrightarrow \begin{cases} 2x+y+1=0 \\ -2x-4y+2=0 \end{cases}$$

$$-3y+3=0 \Rightarrow y=1 \ (+1) \Rightarrow x=-1 \ (+1)$$

The critical point is $(-1,+1)$ or $(-1,+1,-11)$. To classify this point, (+1) compute the discriminant:

$$D = \begin{vmatrix} f_{xx}(-1,1) & f_{xy}(-1,1) \\ f_{xy}(-1,1) & f_{yy}(-1,1) \end{vmatrix} \overset{(+1)}{=} f_{xx}(-1,1) f_{yy}(-1,1) - \left(f_{xy}(-1,1)\right)^2 = (2)(2) - (1)^2 = 3 > 0 \ (+1)$$

Since $D > 0$ and $f_{xx}(-1,1) > 0$, (+1) $(-1,1)$ is a minimum. (+1)

7. (10 points) Suppose there is a particle whose position on a curve is given by a differentiable vector function. If the arc length for $t \geq 0$ is given by $s(t) = \frac{3}{2}t^2$, and the length of the acceleration vector is $|\boldsymbol{a}(t)| = \sqrt{10}$. Please find the curvature $\kappa(s)$ for this curve.

Since $\vec{a}(t) = s''(t)\hat{T}(t) + (s'(t))^2 \kappa(s(t)) \hat{N}(t)$, (+2) the Pythagorean theorem implies that

$$|\vec{a}(t)|^2 = (s''(t))^2 + (s'(t))^4 \kappa^2(s(t)) \quad (+3)$$

Hence given the above information,

$$10 = 3^2 + (3t)^4 (\kappa(s(t)))^2 \quad (+2)$$

$$\Rightarrow \kappa(s(t)) = \frac{1}{9t^2} \quad (+2)$$

But $s(t) = \frac{3}{2}t^2 \Leftrightarrow \left(\frac{2s}{3}\right) = t^2$, thus $\kappa(s) = \frac{1}{9\left(\frac{2s}{3}\right)} = \frac{1}{6s}$ (+1)

Curvature $\kappa(s) =$ $\frac{1}{6s}$

8. (10 points) Suppose that f and g are continuous real-valued functions defined on some domain $D \subset \mathbb{R}^n$. Please prove that the function $f+g$ is also continuous on D.

Given any $\epsilon > 0$, since f is continuous at $(x_0, y_0) \in D$, $\exists\, \delta_1 > 0$ such that $|f(x,y) - f(x_0,y_0)| < \epsilon/2$ whenever $d((x,y),(x_0,y_0)) < \delta_1$. Since g is also continuous at $(x_0,y_0) \in D$, $\exists\, \delta_2 > 0$ such that $|g(x,y) - g(x_0,y_0)| < \epsilon/2$ whenever $d((x,y),(x_0,y_0)) < \delta_2$. **+4**

Define $\delta := \min\{\delta_1, \delta_2\}$. **+1** Then whenever $d((x,y),(x_0,y_0)) < \delta \leq \delta_1, \delta_2$, **+1**

$$
\begin{aligned}
|(f+g)(x,y) - (f+g)(x_0,y_0)| &= |f(x,y) + g(x,y) - f(x_0,y_0) - g(x_0,y_0)| \\
&\leq |f(x,y) - f(x_0,y_0)| + |g(x,y) - g(x_0,y_0)| \quad \textbf{+3} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

Thus by definition, **+1** $(f+g)$ is continuous at $(x_0,y_0) \in D$.

QED

9. (10 points) Suppose that $F : \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, and consider the surface $F(x,y,z) = 0$. Please explain why

$$\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial z} = -1$$

By implicit partial **+1** differentiation and the chain **+1** rule,

$$F(x,y,z) = 0 \Rightarrow \textbf{+5}\ \cancel{F_x(x,y,z)}(0) + F_y(x,y,z)\frac{\partial y}{\partial y} + F_z(x,y,z)\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)} \quad \textbf{+1} \quad \text{provided that } F_z(x,y,z) \neq 0.$$

Similar arguments imply that

$$\frac{\partial y}{\partial x} = -\frac{F_x(x,y,z)}{F_y(x,y,z)} \quad \text{and} \quad \frac{\partial x}{\partial z} = -\frac{F_z(x,y,z)}{F_x(x,y,z)}$$

provided that the denominators are not zero.

Thus

$$\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial z} \ \textbf{+1} = \left(\frac{-\cancel{F_y(x,y,z)}}{\cancel{F_z(x,y,z)}}\right)\left(\frac{-\cancel{F_x(x,y,z)}}{\cancel{F_y(x,y,z)}}\right)\left(\frac{\cancel{-F_z(x,y,z)}}{\cancel{F_x(x,y,z)}}\right) \ \textbf{+1} = \underline{\underline{-1}}$$

QED