MA1034 Theoretical Calculus IV **Name:** Solutions

Final **B Term, 2015**

Show all work needed to reach your answers.

1. (10 points) Consider the vectors $\mathbf{v_1} = \langle 3, -1, -5\rangle$, $\mathbf{v_2} = \langle 1, \alpha, -2\rangle$ and $\mathbf{v_3} = \langle -6, \beta, 10\rangle$. For what value of α is $\mathbf{v_1} \perp \mathbf{v_2}$? For what value of β is $\mathbf{v_1} || \mathbf{v_3}$?

$$V_1 \perp V_2 \overset{(+2)}{\Longleftrightarrow} V_1 \cdot V_2 = 0 = 3 - \overset{(+2)}{\alpha} + 10 \Longleftrightarrow \alpha \overset{(+1)}{=} 13$$

$$V_1 \,||\, V_3 \overset{(+2)}{\Longleftrightarrow} \exists\, c \in \mathbb{R} \text{ s.t. } V_3 = cV_1 \Leftrightarrow \langle -6, \beta, 10\rangle \overset{(+1)}{=} c\langle 3, -1, -5\rangle$$

$$\Longleftrightarrow \underset{(+1)}{c = -2}, \quad \beta = -\underset{(+1)}{c = 2}$$

$\alpha =$ 13 $\beta =$ 2

2. (10 points) For the surface $z = f(x, y) = x^2 - 3xy + 7x - 3y^2 + 8$, please find an equation for the tangent plane at $(1, -1, 16)$.

$$\frac{\partial f}{\partial x}(1,-1) \underset{(+2)}{=} (2x - 3y + 7)\Big|_{(1,-1)} = 12 \qquad \frac{\partial f}{\partial y}(1,-1) \underset{(+2)}{=} (-3x - 6y)\Big|_{(1,-1)} = 3$$

$$z = f(1,-1) + \frac{\partial f}{\partial x}(1,-1)(x-1) + \frac{\partial f}{\partial y}(1,-1)(y+1) \quad (+2)$$

$$= 16 + 12(x-1) + 3(y+1) \quad (+2)$$

$$= 7 + 12x + 3y$$

Tangent Plane: $z = 7 + 12x + 3y$ (+2)

3. (10 points) For the vector function $\mathbf{x}(t) = \langle t + 1, t^2 - t + 2, e^{2t}\rangle$, please find the speed $s'(t) = |\mathbf{v}(t)|$ and the unit tangent vector $\mathbf{T}(t)$.

$$v(t) \overset{(+2)}{=} x'(t) \overset{(+1)}{=} \langle 1, 2t - 1, 2e^{2t}\rangle$$

$$|v(t)| \underset{(+1)}{=} \sqrt{1 + (2t-1)^2 + 4e^{4t}} \underset{(+1)}{=} \sqrt{2 - 4t + 4t^2 + 4e^{4t}}$$

$s'(t) = |\mathbf{v}(t)|$ $\sqrt{2 - 4t + 4t^2 + 4e^{4t}}$ $\mathbf{T}(t) =$ (+3) $\dfrac{v(t)}{|v(t)|}$ (+2)

10

4. (10 points) Suppose that $z = F(u, v, w)$, while $u = f(x, y)$, $v = g(x, y)$ and $w = h(x, y)$. If all these functions are differentiable, what does the chain rule imply about $\dfrac{\partial z}{\partial y}$?

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

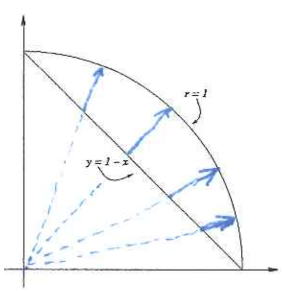
5. (20 points)

10

(a) Please set up the following double integral as an iterated integral in polar coordinates:

$$\iint_D f(r, \theta)\, dA$$

where D is the region shown in the figure below:



$y = 1 - x \Leftrightarrow 1 = x + y = r\cos\theta + r\sin\theta$

$\Leftrightarrow r = \dfrac{1}{\cos\theta + \sin\theta}$ (+3)

(+1) (+1) (+2) (+1) (+1) (+1)

$$I = \int_0^{\pi/2} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} f(r,\theta)\, r\,dr\,d\theta$$

10 (b) For $f(r, \theta) = 1 + 2\sin\theta\cos\theta$, please evaluate this integral.

(+1) (+2)

$$I = \int_0^{\pi/2} (1 + 2\sin\theta\cos\theta) \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} r\,dr\,d\theta = \int_0^{\pi/2} (1 + 2\sin\theta\cos\theta) \left.\frac{r^2}{2}\right|_{\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta$$

(+1) (+2)

$$= \frac{1}{2}\int_0^{\pi/2} (1 + 2\sin\theta\cos\theta)\left(1 - \frac{1}{(\cos\theta+\sin\theta)^2}\right) d\theta = \frac{1}{2}\int_0^{\pi/2} (1 + 2\sin\theta\cos\theta)\left(\frac{(\cos\theta+\sin\theta)^2 - 1}{(\cos\theta+\sin\theta)^2}\right) d\theta$$

(+2) (+2)

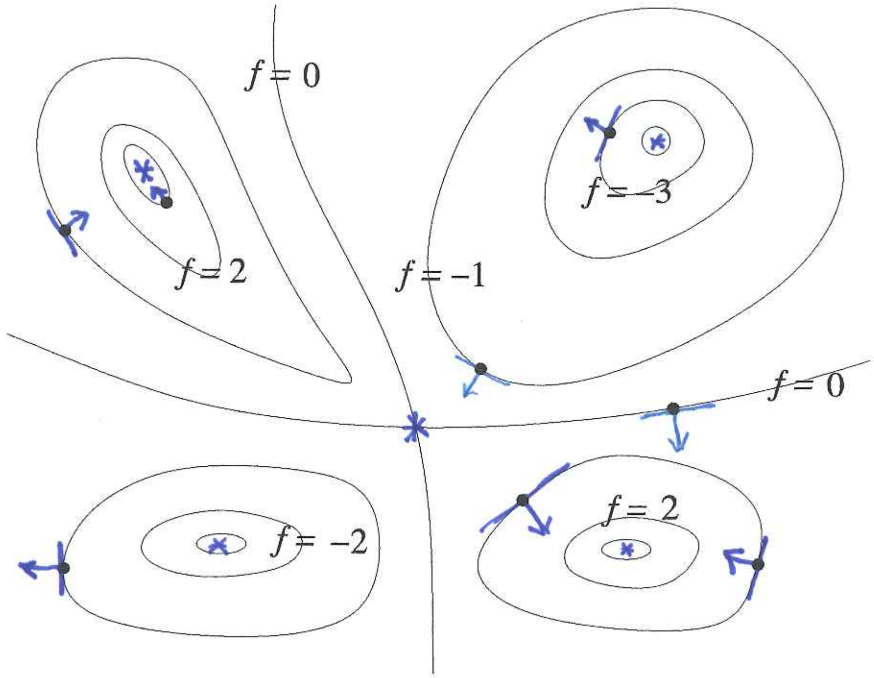
$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \int_0^1 u\, du = 1/2$$

1/2

6. (10 points) In the diagram below, the function f takes on the indicated values on each marked curve. Assuming that f is differentiable, please draw a vector arrow at each point • to show ∇f at that point, then draw a short line segment through each point to indicate the direction in which the directional derivative is zero. Next place an asterisk (*) at each point where the gradient is zero.

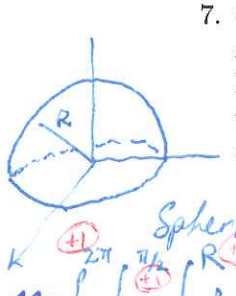


* (+3)
↑ (+3)
– (+3)

Finally, please fill in the following blanks:

- The curves in this diagram are called level curves for the function f.
- The one point where the curves cross is a saddle point (+1).

7. (15 points) 9 Consider a hemispherical dome with radius R sitting on top of the x,y-plane: $x^2+y^2+z^2=R^2$, $z \geq 0$. Suppose that the dome is filled with a gas whose density decreases linearly with height (so $\delta(z)=\delta_0(1-z/R)$). Please set up **but do not evaluate** an iterated triple integral in spherical or cylindrical coordinates (your choice) which represents the mass of this dome.



Spherical

$$M=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{R}\delta_0\left(1-\frac{\rho\cos\varphi}{R}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\delta_0\int_0^{2\pi}d\theta\int_0^{\pi/2}\sin\varphi\int_0^{R}\rho^2\left(1-\frac{\rho\cos\varphi}{R}\right)d\rho\,d\varphi$$

Cylindrical (+2)

$$M=\int_0^{2\pi}\int_0^{R}\int_0^{\sqrt{R^2-r^2}}\delta_0\left(1-\frac{z}{R}\right)dz\,r\,dr\,d\theta$$

$$=\delta_0\int_0^{2\pi}d\theta\int_0^{R}r\int_0^{\sqrt{R^2-r^2}}\left(1-\frac{z}{R}\right)dz\,dr$$

Integral: ____________________

10

8. (10 points) If **a** and **b** are nonzero vectors, please explain in **one sentence** why $\mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = 0$.

$a \times b \cdot a = (a \times b) \cdot a$; since $a \times b$ is perpendicular to both a and b, $(a \times b) \cdot a = 0$ (dot product of perpendicular vectors is zero).

9. (10 points) Consider the function $f(r, \theta) = 2\cos\theta \sin\theta$ for $r \neq 0$, with $f(0, \theta) = 0$. Is this function continuous at the origin (when $r = 0$)? Does $\dfrac{\partial f}{\partial r}$ exist at the origin? Please explain your answers.

+1

Since $\lim_{r \to 0} f(r, \theta)$ would depend on θ, this limit DNE, hence f is not continuous at the origin.

+2 +2

$$\frac{\partial f}{\partial r}(0,\theta) \overset{+1}{:=} \lim_{h \to 0} \frac{f(0+h, \theta) - f(0,\theta)}{h}$$

$$\overset{+1}{=} \lim_{h \to 0} \frac{f(h, \theta)}{h} \overset{+1}{=} \lim_{h \to 0} \frac{2\cos\theta \sin\theta}{h}$$

+1 +1

but again this limit DNE. So $\frac{\partial f}{\partial r}$ DNE at the origin.