MA1034 Theoretical Calculus IV Name: Solutions

Final **B Term, 2018**

Show all work needed to reach your answers.

1. (10 points) For $\boldsymbol{u} = \langle 1, 9, -1\rangle$ and $\boldsymbol{v} = \langle -3, 1, \alpha\rangle$, please find the value of α which make these two vectors orthogonal.

$$\vec{u} \perp \vec{v} \overset{+5}{\Leftrightarrow} \vec{u}\cdot\vec{v} = 0 \overset{+3}{\Leftrightarrow} -3 + 9 - \alpha = 0 \overset{+2}{\Leftrightarrow} \alpha = 6$$

$\alpha =$ 6

2. (15 points) For the surface $xyz = 6$, please find an equation for the tangent plane at the point $P_0(1, -2, -3)$.

Here $\vec{x}_0$ $F(x,y,z) = xyz - 6 = 0 \overset{+4}{\Rightarrow} \vec{\nabla}F(x,y,z) = \langle yz, xz, xy\rangle$

So $\vec{N} = \vec{\nabla}F(1,-2,-3) \overset{+4}{=} \langle 6, 3, 2\rangle$, and the equation is

$\vec{N}\cdot(\vec{x}-\vec{x}_0) = 0$ (+4) $6(x-1) - 3(y+2) - 2(z+3) = 0$

Either (+3)

Equation: $6x - 3y - 2z = 18 \Leftrightarrow z = 3x - \frac{3}{2}y - 9$

3. (15 points) Please compute $\iiint_\Omega z\, dV$ where Ω is hemisphere under $x^2 + y^2 + z^2 = 4$ and above $z = 0$.

In spherical coordinates, $\iiint_\Omega z\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (\rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$ (+1) (+2) (+1) (+1) (+2)

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2}\sin\varphi\cos\varphi\, d\varphi\right)\left(\int_0^2 \rho^3\, d\rho\right) = (2\pi)\left(\int_0^1 u\, du\right)\left(\frac{\rho^4}{4}\Big|_0^2\right) = (2\pi)\left(\frac{1}{2}\right)(4)$$

$u = \sin\varphi$
$du = \cos\varphi\, d\varphi$

(+2) (+2) (+3) (+3)

In cylindrical coordinates, $\iiint_\Omega z\, dV = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} z\,(dz\, r\, dr\, d\theta)$ (+1) (+1) (+2) (+2)

$$= \left(\int_0^{2\pi} d\theta\right)\left(\frac{1}{2}\int_0^2 z^2\Big|_0^{\sqrt{4-r^2}}\, dr\right) = \frac{1}{2}(2\pi)\int_0^2 (4-r^2)\,r\, dr = \pi\left(\frac{4r^2}{2} - \frac{r^4}{4}\Big|_0^2\right)$$

(+2) (+2) (+2) (+2)

$$= \pi(8-4) = 4\pi$$ (+1)

4π

4. (10 points) For $f(x, y) = x^3 + 5xy^2 - y^3$, please compute the directional derivative $D_{\boldsymbol{u}} f(2,1)$ in the direction of $\langle 1, 1 \rangle$.

Here $\hat{u} = \frac{\langle 1,1 \rangle}{\sqrt{2}}$ (+1), so

(+3) (+3)

$$D_{\hat{u}} f(2,1) = \nabla f(2,1) \cdot \hat{u} = \langle 3x^2 + 5y^2, 10xy - 3y^2 \rangle \Big|_{(2,1)} \cdot \frac{\langle 1,1 \rangle}{\sqrt{2}}$$

$$= \langle 17, 17 \rangle \cdot \frac{\langle 1,1 \rangle}{\sqrt{2}} = \frac{2(17)}{\sqrt{2}} = 17\sqrt{2}$$

(+1) (+2)

$17\sqrt{2}$

5. (15 points) If the partial derivative exists, please compute $f_y(0, 0)$ for

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_y(0,0) := \lim_{k \to 0} \frac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{\frac{-k^3}{k^2}}{k} = \lim_{k \to 0} -1 = -1$$

(+6) (+6) (+2)

(+1)

-1

6. (15 points) Set up but **do not evaluate** an iterated integral equal to

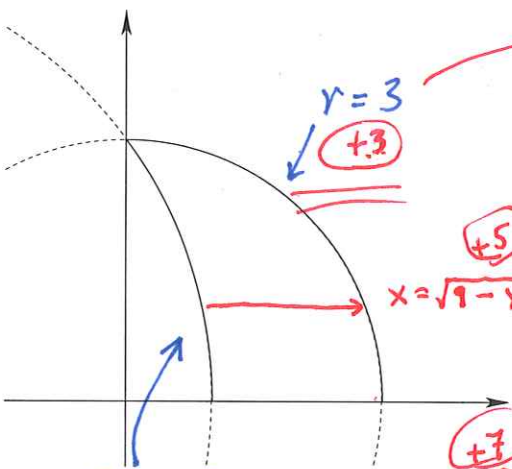
$$\iint_D f(x,y)\, dA$$

where D is the domain in the first quadrant between the circles $(x+4)^2+y^2=25$ and $x^2+y^2=9$.

In polar coordinates

$$\iint_D f(x,y)\,dA = \overset{(+2)}{\int_{\underset{(+1)}{0}}^{\pi/2}} \int_{-4\cos\theta+\sqrt{16\cos^2\theta+9}}^{3} \overset{(+1)}{f(r\cos\theta, r\sin\theta)}\, \overset{(+2)}{r\,dr\,d\theta}$$

$r=3$ (+3)



$x=\sqrt{9-y^2}$ (+5)

$y=\sqrt{25-(x+4)^2}$ $(x+4)^2+y^2=25 \Rightarrow x=-4+\sqrt{25-y^2}$ (+7)

only +

$x^2+8x+16+y^2=25$

$(x^2+y^2)+8x=9$

$r^2+8r\cos\theta=9$

$\Rightarrow r=\dfrac{-8\cos\theta \pm \sqrt{64\cos^2\theta-4(-9)}}{2}$

$=-4\cos\theta+\sqrt{16\cos^2\theta+9}$ (+6)

$r>0 \Rightarrow +$ only

Form: (+1)

$$(+2)\int_0^3\left(\int_{-4+\sqrt{25-y^2}}^{\sqrt{9-y^2}} f(x,y)\,dx\right)dy$$

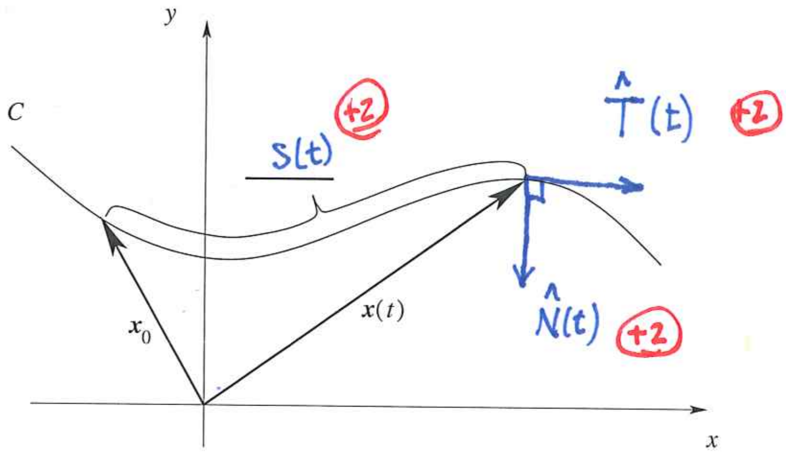
(+5)

$$(+1)\int_0^1\int_{\sqrt{25-(x+4)^2}}^{\sqrt{9-x^2}} f(x,y)\,dy\,dx + (+1)\int_1^3\int_0^{\sqrt{9-x^2}} f(x,y)\,dy\,dx$$

(+7) Form: (+1)

$$\int_0^{\pi/2}\int_{-4\cos\theta+\sqrt{16\cos^2\theta+9}}^{3} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

7. (20 points) Suppose $\boldsymbol{x}(t)$ is a differentiable vector function which traces out the curve C, and suppose $\boldsymbol{x}'(t) \neq 0$ for any t:



(a) Please correctly fill in the blank on the diagram above, and also correctly locate and draw the unit tangent vector $\boldsymbol{T}(t)$ and the unit normal vector $\boldsymbol{N}(t)$.

(b) What are the definitions of $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$.

$$\boldsymbol{T}(t) = \frac{\vec{V}(t)}{|\vec{V}(t)|} = \frac{\vec{x}'(t)}{|\vec{x}'(t)|} \quad +3 \qquad \boldsymbol{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \quad +3$$

(c) Based on these definitions, please explain why $\boldsymbol{T}(t) \perp \boldsymbol{N}(t)$, that is, prove that $\boldsymbol{T}(t) \perp \boldsymbol{N}(t)$.

Since $1 = |\hat{T}(t)|^2 = \hat{T}(t) \cdot \hat{T}(t) \ \forall t$, then

+8

$$0 = \frac{d}{dt}(1) = \frac{d}{dt}(\hat{T} \cdot \hat{T}) = 2\,\hat{T}(t) \cdot \vec{T}'(t)$$

Thus $\hat{T}(t) \perp \vec{T}'(t)$ and hence $\hat{T}(t) \perp \vec{N}(t)$.