MA1034 Theoretical Calculus IV Name: Solutions

High: 92
Median: 81
Low: 69

Final **B Term, 2019**

This is a closed book/notes test; no electronic devices allowed. Show all work needed to reach your answers. There are 10 free points on this exam.

1. (10 points) If $u(x, y) = x^2 \cos 7y + 3x^5 \sin 4y$, please find the gradient, ∇u.

Signs: +1 Form: +5

$$\nabla u(x, y) = \langle 2x\cos 7y + 15x^4 \sin 4y,\ -7x^2 \sin 7y + 12x^5 \cos 7y \rangle$$

+1 +1 +1 +1

2. (10 points) Suppose that $u = u(x, y, z)$ and that $x = f(s, t) = st^2$, $y = g(s, t) = 3s + 2t$ and $z = h(s, t) = 4t - s$. Please write $\partial u/\partial t$ as explicitly as possible.

+1 +5 Form

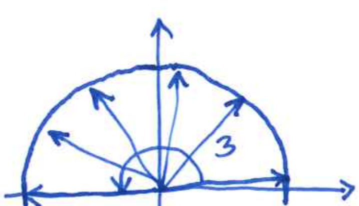
By the chain rule: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$

$$= 2st\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + 4\frac{\partial u}{\partial z}$$

+1 +1 +1

$$\frac{\partial u}{\partial t} = 2st\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + 4\frac{\partial u}{\partial z}$$ +1

3. (15 points) Please compute $\iint_D y\, dA$ where D is semicircular disk centered at $(0, 0)$ of radius 3 with $y > 0$.



$$\iint_D y\, dA = \int_0^{\pi}\left(\int_0^3 (r\sin\theta)(r\,dr)\,d\theta\right)$$ +5

$$= \left(\int_0^{\pi} \sin\theta\, d\theta\right)\left(\int_0^3 r^2\, dr\right)$$ +1 +1

$$= \left(-\cos\theta\Big|_0^{\pi}\right)\left(\frac{r^3}{3}\Big|_0^3\right)$$ +2 +2

$$= (\cos 0 - \cos(\pi))\left(\frac{3^3}{3}\right) = 2 \cdot 9$$ +2 +1

+1

18

4. (20 points) For $t > 0$, consider the vector function $\boldsymbol{x}(t) = \langle t^2/2+3, t^3/3+1\rangle$ which gives the position of a particle moving in the x_1, x_2-plane. Please find the velocity vector $\boldsymbol{v}(t)$, the speed $\dot{s}(t)$, the arc length $s(t)$, and the unit tangent vector $\boldsymbol{T}(t)$. Use $t_0 = 0$, $x_0 = \langle 3, 1\rangle$.

① $\vec{v}(t) = \dot{\vec{x}}(t) = \langle t, t^2 \rangle$ +2 +3

② $\dot{s}(t) = |\vec{v}(t)| = \sqrt{t^2 + (t^2)^2} = t\sqrt{1+t^2}$ (since $t > 0$). +1 +3 +1

③ $s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau = \int_0^t \tau\sqrt{1+\tau^2}\, d\tau = \frac{1}{2}\int_1^{1+t^2} \sqrt{u}\, du$ +1 +1 +1

let $u = 1+\tau^2 \Rightarrow du = 2\tau\, d\tau$

$= \frac{\frac{1}{2}}{\frac{3}{2}} \left(u^{3/2} \right|_1^{1+t^2} = \frac{1}{3}\left(1+t^2\right)^{3/2} - \frac{1}{3}$ +1 +1

④ $\hat{T}(t) = \frac{\langle t, t^2 \rangle}{t\sqrt{1+t^2}} = \frac{\langle 1, t \rangle}{\sqrt{1+t^2}}$ +4 +1

$\boldsymbol{v}(t) =$ $\langle t, t^2 \rangle$

$\dot{s}(t) =$ $t\sqrt{1+t^2}$

$s(t) =$ $\frac{(1+t^2)^{3/2} - 1}{3}$

$\boldsymbol{T}(t) =$ $\frac{\langle 1, t \rangle}{\sqrt{1+t^2}}$

5. (10 points) Please compute the following limit, or explain why the limit does not exist:

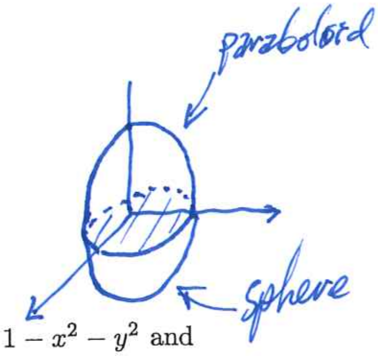
$$L = \lim_{\substack{(x,y) \to (-2,2) \\ y \neq -x}} \frac{x+y}{x^2 - y^2}$$

$= \lim_{\substack{(x,y)\to(-2,2) \\ y \neq x}} \cancel{\frac{x+y}{(x+y)(x-y)}}$ Factor & cancel +7 $= \lim_{(x,y)\to(-2,2)} \frac{1}{x-y}$ +2

$= \frac{1}{-2-2}$

+1

$L =$ $-1/4$



6. (15 points) Set up but **do not evaluate** an iterated integral equal to

$$\iiint_{\Omega} f(x, y, z)\ dV$$

where Ω is the egg bounded above the x, y-plane by the paraboloid $z = 1 - x^2 - y^2$ and bounded below the x, y-plane by the unit sphere $x^2 + y^2 + z^2 = 1$

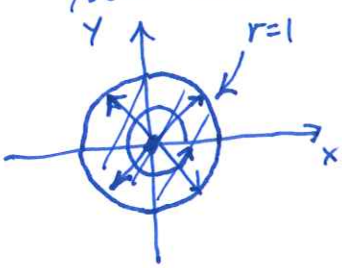
Best set up in cylindrical coordinates:

Upper Surface: $z = 1 - x^2 - y^2 = 1 - r^2$ (paraboloid).

Lower Surface: $z = -\sqrt{1 - x^2 - y^2} = -\sqrt{1 - r^2}$ (lower hemisphere).

$$\iiint_{\Omega} f(x,y,z)\,dV = \int_0^{2\pi}\left[\int_0^1 \left[\int_{-\sqrt{1-r^2}}^{1-r^2} f(r\cos\theta, r\sin\theta, z)\,dz\right] r\,dr\right] d\theta$$

The portion of the x,y-plane supporting the egg is the unit circular disk: for $z = 0$, either surface has $x^2 + y^2 = r^2 = 1 \Rightarrow r = 1$



+5 +5 +1 +1 +1 +1 +1

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r^2} f(r\cos\theta, r\sin\theta, z)\,dz\; r\,dr\; d\theta$$

7. (10 points) Suppose three planes are randomly arranged in $\mathbb{R}^3$ (so that in particular none of the normal vectors of the three planes are parallel, nor can these normal vectors themselves lie in any single plane). Please describe the intersections of these three planes. Please carefully prove your answer.

From the general equation for a plane in $\mathbb{R}^3$, the three planes are

$$\begin{aligned} A_1 x + B_1 y + C_1 z &= D_1 \\ A_2 x + B_2 y + C_2 z &= D_2 \\ A_3 x + B_3 y + C_3 z &= D_3 \end{aligned} \qquad (1)$$

+2

and the normal vectors are

$$\vec{N}_1 = \langle A_1, B_1, C_1 \rangle$$
$$\vec{N}_2 = \langle A_2, B_2, C_2 \rangle$$
$$\vec{N}_3 = \langle A_3, B_3, C_3 \rangle$$

+2

Because these 3 vectors do not lie in any single plane, it is not possible to write any $\vec{N}$ as a linear combination of the other two:

$$\vec{N}_K \neq \alpha_K \vec{N}_{K+1} + \beta_K \vec{N}_{K+2}$$

+2 This means that the three equations in three unknowns (1) have a unique solution $(x, y, z) = (x_0, y_0, z_0)$, implying that the three planes must intersect in a single point: +2 (x_0, y_0, z_0). The planes then intersect pairwise in three +2 distinct lines that all pass through (x_0, y_0, z_0). So in $\mathbb{R}^3$