MA1971 Bridge to Higher Math   Name: Solutions

Final   D Term, 2013

Scale
A: 86-100
B: 74-85
C: 63-73
D: 55-62
F: 0-54

Show all work needed to reach your answers.

1. (20 points) Consider the implication $A \Rightarrow B$ where A and B are themselves statements. For this implication, please state the following:

5 each

(a) contrapositive: $\neg B \Rightarrow \neg A$

(b) negation: $A \wedge \neg B$

(c) direct implication: $A \Rightarrow B$

(d) converse: $B \Rightarrow A$

2. (20 points) THEOREM: If a sequence $\{a_n\} \subseteq \mathbb{R}$ converges, then the limit is unique.

Proof: Suppose that $a_n \to a$ for some $a \in \mathbb{R}$ and that $a_n \to \alpha$ for some $\alpha \in \mathbb{R}$. To prove that the limit is unique, one must show that $a = \alpha$ 2. So, given any $\epsilon > 0$ 2, there exists $N \in \mathbb{Z}^+$ 2 such that $|a_n - a|$ 2 $< \epsilon/2$ and also $|a_n - \alpha|$ 2 $< \epsilon/2$ for all $n > N$ 2. By the triangle inequality,

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$$|a - \alpha| \leq |a - a_n| + |a_n - \alpha| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Since ϵ is arbitrarily small, $a = \alpha$ 2 and thus the limit is unique.

□

3. (15 points) Let $f : A \to B$ be a function, and let $S \subseteq B$ and $T \subseteq B$. Please prove that $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

Let $x \in f^{-1}(S \cap T)$. Then $f(x) \in S \cap T$, and thus $f(x) \in S$ and $f(x) \in T$. So $x \in f^{-1}(S)$ and $x \in f^{-1}(T) \Rightarrow$ $x \in f^{-1}(S) \cap f^{-1}(T)$. All of this implies $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.

4. (15 points) Please show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Hint: Induction.

- For $n=1$, $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$, so the formula works for this base case.
- Now assume the formula holds for $n=k$; we must show it holds for $n=k+1$. Notice that

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \overset{\boxed{\text{Inductive Assumption}}}{=} \frac{k(k+1)}{2} + (k+1) = \left(\frac{k}{2} + 1\right)(k+1)$$

$$= \frac{(k+2)(k+1)}{2} = \frac{(k+1)((k+1)+1)}{2} = \frac{n(n+1)}{2}$$

Thus by induction, the formula holds $\forall n \in \mathbb{Z}^+$.

5. (15 points) Please negate the following statement about a sequence $\{a_n\}$ and its limit a:

$$\forall N \in \mathbb{Z}^+,\ \exists\, \epsilon > 0 \text{ s.t. } |a_n - a| < \epsilon\ \forall\, n > N$$

Negation: $\exists\, N \in \mathbb{Z}^+$ s.t. $\forall\, \epsilon > 0$, $|a_n - a| \geq \epsilon$ for some $n > N$.

6. (15 points) Consider the sum of any six consecutive positive integers. What can said about the divisors (factors) of this sum? Please prove your answer.

Let $S = n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5)$ be the sum of six consecutive positive integers (depending on $n \in \mathbb{Z}^+$). So $S = 6n + 15 = 3(2n+5)$ and thus S is the product of two factors: 3 and an odd integer, 7 or greater.