MA1971 **Bridge to Higher Math** Name: Solutions

Final **D Term, 2015**

Show all work needed to reach your answers.

1. (18 points) Several short-answer questions:

   (a) What do each of the following symbols mean? (Please write meaning after symbol.)

   +2 +2 +2

   $\exists$ "there exists" or "for some" $\forall$ "for all" $\exists!$ "there exists uniquely"

   (b) What is the meaning of $\aleph_0$ (aleph naught)?

   This is the cardinality of $\mathbb{Z}_+$ and thus of any countably infinite set. +6

   (c) If $x \in \mathcal{P}(A)$ what is the relationship between x and A? ($\mathcal{P}(A)$ is the power set of A.)

   $x \subset A$ +6

2. (17 points) Suppose that U is some universal set, and suppose that $A,B \subset U$. Please prove that $A^c \cap B \subset (A \cup B^c)^c$. +3

   Suppose $x \in A^c \cap B$. Then $x \in A^c$ and $x \in B$. +3 So $x \notin A$ +3 and $x \notin B^c$; thus $x \notin A \cup B^c$. +3 Finally this implies $x \in (A \cup B^c)^c$, +3 establishing that $A^c \cap B \subset (A \cup B^c)^c$. +2

   QED

3. (20 points) Which of the following sequences (a) **must converge**, (b) which **must diverge**, and (c) which **might** converge or **might** diverge?

4 each

(a) A decreasing sequence of positive real numbers.

It's bounded below by zero.

must converge

(b) An oscillating sequence of real numbers whose oscillation amplitude decreases to 1.

If a sequence oscillates with amplitude approximately 1, it can not converge.

must diverge

(c) An increasing sequence of real numbers.

might converge; might diverge

(d) A decreasing sequence of rational numbers whose greatest lower bound is a negative real number.

It must converge to its glb.

must converge

(e) A sequence of real numbers which is bounded both above and below.

might converge; might diverge

4. (15 points) Suppose that the sum of the digits of a number $n \in \mathbb{Z}$ is divisible ~~my~~ b 9. Please show that $9|n$. Hint: If n has $N+1$ digits, let

$$n = a_N a_{N-1} ... a_2 a_1 a_0 = \sum_{k=0}^{N} a_k 10^k$$

Notice that $n = \sum_{k=0}^{N} a_k 10^k = a_0 + 10a_1 + 100a_2 + \ldots + 10^N a_N$

$= (a_0 + a_1 + \ldots + a_N) + 9a_1 + 99a_2 + \ldots + \underbrace{999\ldots9}_{N} a_N$ +6

So if $9 \mid \sum_{k=0}^{N} a_k$, since $9|9, 9|99, \ldots, 9|\underbrace{999\ldots9}_{N}$, then +3 +3

+3 $9|n$. QED

5. (15 points) Please give the contrapositive of the following statement:

If $x > 0$ but $y < 0$, then $z = 0$ or $q \in \mathbb{Q}$.

Contrapositive: $\underbrace{\text{If } z \neq 0 \text{ and } q \notin \mathbb{Q}, \text{ then } x \leq 0 \text{ or } y \geq 0.}_{\text{Format: } +3}$

+2 +2 +2 +2 +2 +2

6. (15 points) A graph which is the union of disconnected trees is sometimes called a forest. Suppose that $T := \{T_1, T_2, ..., T_k, ...\}$ be a set of trees, and suppose that

$$F := \bigcup_{k=1}^{|T|} T_k$$

is such a forest (here $|T|$ be the number of trees in the forest). Let $|V_k|$ and $|E_k|$ be the numbers of vertices and edges in the k-th tree, T_k; let $|V|$ and $|E|$ be the total numbers of vertices and edges overall in the forest. Please find a formula relating $|T|$, $|V|$ and $|E|$, ~~and~~ also starting from the Euler formula, please prove that your formula is correct.

For each tree T_k, the Euler formula is $|V_k| - |E_k| + |R_k| = 2$. +4

But for any tree, $|R_k| = 1$ (there's only one region), so $|V_k| - |E_k| = 1$ +4. Since this formula holds for each tree,

+4

$$|V| - |E| = \sum_{k=1}^{|T|} |V_k| - \sum_{k=1}^{|T|} |E_k| = \sum_{k=1}^{|T|} (|V_k| - |E_k|) = \sum_{k=1}^{|T|} 1 = |T|$$

So the desired formula is simply $|V| - |E| = |T|$. +3

QED