MA1971 Name: Solutions

Quiz 5 **D Term, 2015**

High: 20
Median: 17
Low: 7

Show all work needed to reach your answers.

1. (10 points) Your friend is a something-not-mathematics major, and tells you that it is possible to count the real numbers in $[0, 1]$ by writing them in binary form (so each number $x_j \in [0, 1]$ is of the form $0.x_{j1}x_{j2}x_{j3}...$ where x_{jk} is either 0 or 1). Your friend gives the count as

   $x_1 = 0.x_{11}x_{12}x_{13}x_{14}...$
   $x_2 = 0.x_{21}x_{22}x_{23}x_{24}...$
   $x_3 = 0.x_{31}x_{32}x_{33}x_{34}...$
   $x_4 = 0.x_{41}x_{42}x_{43}x_{44}...$
   and so forth

   Is your friend correct or wrong? Please explain why. +1

   Consider the number $b := 0.b_1b_2b_3\ldots$ (also written in binary) where $b_j := \begin{cases} 0 & x_{jj} = 1 \\ 1 & x_{jj} = 0 \end{cases}$. +4 So $b \in [0,1]$, +1 +1 while $b \neq x_k \ \forall k \in \mathbb{Z}^+$, implying that b is not on the list. So my friend is wrong. +3

2. (10 points) By the greatest lower bound property of $\mathbb{R}$ (the real numbers), a decreasing sequence of real numbers that is bounded below must converge to a limit L which is also a real number. Does a decreasing sequence of rational numbers always converge to a rational number? Please give a counterexample to show that $\mathbb{Q}$ (the rational numbers) has no such greatest lower bound property. Hint: Think about the examples we have discussed in class and the homework exercises.

   Other examples also works.

   Consider the sequence $\left\{ \frac{1}{3}, \frac{1}{3+1/3}, \frac{1}{3+\frac{1}{3-1/3}}, \frac{1}{3+\frac{1}{3-\frac{1}{3-1/3}}}, \ldots \right\}$ +3 (the first sign in this continued fraction is positive; all the rest are negative). Each element in this sequence is rational, and we showed that it is decreasing +2 and bounded below by zero +2 (among other bounds), so it must converge, +1 but not necessarily to a rational number. So letting $x = \frac{1}{3-x} \Leftrightarrow x^2 - 3x + 1 = 0 \Leftrightarrow x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3}{2} - \frac{\sqrt{5}}{2}$, and thus the limit is $L = \frac{1}{3+x} = \frac{2}{9-\sqrt{5}} \cdot \frac{9+\sqrt{5}}{9+\sqrt{5}} = \frac{9+\sqrt{5}}{38}$ +2