Bridge to Higher Mathematics, MA 1971 D21

Exercise Set I

1. Please give an example of a predicate $\mathbf{A}(x)$ for which "For all $x \in \mathbb{R}$, $\mathbf{A}(x)$" is true. Then give a separate example of a predicate $\mathbf{B}(x)$ for which "For all $x \in \mathbb{R}$, $\mathbf{B}(x)$" is false, but "There exists $x \in \mathbb{R}$ such that $\mathbf{B}(x)$" is true.

2. Please identify the hypotheses and conclusions in each implication. Then decide which statements are true and which are false.

   a. For $x, y, z \in \mathbb{Z}^+$, if $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.

   b. If x is an integer, then $x^2 \geq x$.

   c. For $x \in \mathbb{R}$, if $x^2 > 11$, then x is positive.

   d. If f is a polynomial of odd degree, then f has at least one real root.

   e. If x is an integer, then $x^3 \geq x$.

3. Create a truth table to verify that each of the following is a tautology.

   a. $(\mathrm{A} \wedge (\mathrm{A} \Rightarrow \mathrm{B})) \Longrightarrow \mathrm{B}$

   b. $(\mathrm{A} \Rightarrow (\mathrm{B} \wedge \mathrm{C})) \Longrightarrow (\mathrm{A} \Rightarrow \mathrm{B})$

   c. $((\mathrm{A} \Rightarrow \mathrm{B}) \wedge (\mathrm{B} \Rightarrow \mathrm{C})) \Longrightarrow (\mathrm{A} \Rightarrow \mathrm{C})$

   d. $(\mathrm{A} \Rightarrow (\mathrm{B} \vee \mathrm{C})) \Longleftrightarrow ((\mathrm{A} \wedge \neg\mathrm{B}) \Rightarrow \mathrm{C})$

4. Construct a truth table to show that it is possible for $\mathrm{A} \Rightarrow \mathrm{B}$ to be true while its converse $\mathrm{B} \Rightarrow \mathrm{A}$ is false.

5. There are some useful rephrasings that involve negation. Construct a truth table to compare the truth values of the following four statements:

$$\neg(\mathrm{A} \wedge \mathrm{B}) \qquad \neg\mathrm{A} \wedge \neg\mathrm{B} \qquad \neg(\mathrm{A} \vee \mathrm{B}) \qquad \neg\mathrm{A} \vee \neg\mathrm{B}$$

   Which pairs are equivalent?

6. Rephrase the statement "x is not greater than 7" in positive terms.

7. Negate the following predicates. Write each negation as positively as possible.

   a. The roots of a polynomial $P(x)$ are either all real or all genuinely complex numbers.

   b. For $x \in \mathbb{R}$, both $x < 0$ and x is irrational.

   c. For $x, y, z \in \mathbb{Z}^+$, both $x + y$ and $y + z$ are even.

8. Negate the following statements. Write each negation as positively as possible.

    a. There exists an odd prime number.
    b. For all real numbers x, $x^3 = x$.
    c. Every positive integer is the sum of distinct powers of three.
    d. There exists a positive real number y such that for all real numbers x, $y^2 = x$.

9. Negate the following statements. Write each negation as positively as possible. Which statements or true and which are false.

    a. If x is an odd integer, then x^2 is an even integer.
    b. If f is a continuous function, then f is a differentiable function.
    c. If f is a differentiable function, then f is a continuous function.
    d. If f is a polynomial with integer coefficients, then f has at least one real root.

10. Give counterexamples to the following false statements.

    a. If a real number is greater than 5, then it is less than 10.
    b. If x is a real number, then $x^3 = x$.
    c. All prime numbers are odd numbers. *What is the hypothesis here, and what is the conclusion?*

11. Use a direct proof to show that "If $x + y$ is even and $y + z$ is even, then $x + z$ is even."

12. Find the contrapositives of the following statements. Write things in positive terms whenever possible.

    a. If $x < 0$, then $x^2 > 0$.
    b. If $x \neq 0$, then there exists y for which $xy = 1$.
    c. If x is an even integer, then x^2 is an even integer.
    d. If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.
    e. If f is a polynomial of odd degree, then f has at least one real root.

13. Let A, B, Q and P be statements. Construct a truth table to show that the following statements are equivalent:

$$\mathrm{Q} \qquad \text{and} \qquad (\neg \mathrm{Q}) \Rightarrow (\mathrm{P} \wedge \neg \mathrm{P})$$

14. Use proof by contradiction to show that "If x is an integer, then x cannot be both even and odd."