Show your work in the space provided. Unsupported answers will not receive full credit.

1. Find the general solutions of the following differential equations. Where initial conditions are given also solve the particular initial value problem.

   (a) \( y'' + 2y' + 2y = 0 \)
   
   \[ \text{char. eq: } r^2 + 2r + 2 = 0 \quad r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i \]
   
   \[ Y_h = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x) \]

   (b) \( y'' - 4y' + 4y = 0 \) with \( y(0) = 2, y'(0) = 4 \)

   \[ \text{char. eq: } r^2 - 4r + 4 = 0, \quad (r-2)^2 = 0 \]
   
   \[ r = 2, \text{ double root} \]
   
   \[ Y_h = c_1 e^{2x} + c_2 xe^{2x} \]

   So
   
   \[ Y_h' = 2c_1 e^{2x} + c_2 (2xe^{2x} + e^{2x}) \]
   
   \[ Y_h(0) = c_1 = 2 \]
   
   \[ Y_h'(0) = 2c_1 + c_2 = 4, \quad c_2 = 0 \]
   
   \[ Y_h = 2e^{2x} \]
2. Show that the following DE is exact, and find the solution.

\[(1 + e^x y + x ye^x) \, dx + (xe^x + 2) \, dy = 0\]

\[\frac{\partial u}{\partial y} \left( 1 e^x y + x ye^x \right) = e^x + xe^x\]

\[\frac{\partial u}{\partial x} (xe^x + 2) = e^x + xe^x\]

They are equal, so the DE is exact.

Know:

\[\frac{\partial u}{\partial x} = (x e^x y + x ye^x)\]

\[\frac{\partial u}{\partial y} = xe^x + 2\]

Integrate \(\frac{\partial u}{\partial y}\) w.r.t. \(y\), get:

\[u(x, y) = x ye^x + 2y + \Psi(x)\]

So

\[\frac{\partial u}{\partial x} (x, y) = ye^x + x ye^x + \Psi'(x) = 1 + e^x y + x ye^x\]

So \(\Psi'(x) = 1\), \(\Psi(x) = x\)

So

\[u(x, y) = x ye^x + 2y + x\]

Solutions satisfy

\[xye^x + 2y + x = C\]
3. Consider the following differential equation

\[ y' = \frac{x + 4}{y - 1} \]

Describe the points of the plane where our basic existence and uniqueness result does not guarantee a unique solution of the differential equation. Do not solve the DE.

\[ F(x, y) = \frac{x + 4}{y - 1} \]

\[ F_y(x, y) = -\frac{(x + 4)}{(y - 1)^2} \]

\( F \) and \( F_y \) are continuous except on the line \( y = 1 \), so our Existence and Uniqueness Theorem applies to any point \( (a, b) \) with \( b \neq 1 \).
4. Answer the following questions about linear, first order differential equations.

(a) Write down the general form for a first-order, linear differential equation.
\[ y' + p(x)y = g(x) \]

(b) Write down the general form for a first-order, linear, homogeneous differential equation.
\[ y' + p(x)y = 0 \]

(c) Use separation of variables to find a formula for the general solution of a first-order, linear, homogeneous differential equation.

\[ y' = -p(x)y \]

Separate variables, get

\[ \frac{1}{y} \, dy = -p(x) \, dx \quad (y \neq 0, \text{ but } y=0 \text{ is a soln}) \]

so

\[ \ln |y| = -\int p(x) \, dx + C \]

so

\[ |y| = e^{-\int p(x) \, dx} \cdot e^C \]

Replace \( e^C \) with \( k \), which is allowed to be positive or negative, or zero, and you get

\[ y = k e^{-\int p(x) \, dx} \]

Notice that \( k=0 \) gives \( y=0 \), so we don't have to list it separately.
5. Consider the following differential equation.

\[ y' + \cos(x)y = 3\cos(x) \]

(a) Find the general solution to this differential equation.

\[ p(x) = \cos(x) \quad \int p(x) \, dx = \sin(x), \text{ so we have} \]

\[
\begin{aligned}
(e^y)' &= 3\cos(x) \cdot e^y \\
\Rightarrow e^y &= 3 \int \cos(x) e^y \, dx \\
&= 3 \sin(x) + C \\
\Rightarrow y &= 3 + C e^{-\sin(x)}
\end{aligned}
\]

(b) Use your general solution to solve the IVP \( y' + \cos(x)y = 3\cos(x), \ y(0) = 4 \).

\[
\begin{aligned}
y(0) &= 3 + C e^{0} = 3 + C = 4, \quad C = 1 \\
\Rightarrow y &= 3 + e^{-\sin(x)}
\end{aligned}
\]

(c) Without solving the differential equation, write down the general solution to

\[ y' + \cos(x)y = 9\cos(x) \]

\[
\begin{aligned}
q\cos(x) &= 3 \cdot 3\cos(x), \text{ so solution is} \\
&= -\sin(x) \\
\Rightarrow y &= 9 + Ce^{-\sin(x)}
\end{aligned}
\]
6. Consider the following differential equation.

\[ y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0 \]

First, show that \( y = x \) and \( y = x^2 \) are solutions for \( x > 0 \). Then, use the Wronskian to show that the general solution to this equation for \( x > 0 \) has the form

\[ y = c_1 x + c_2 x^2 \]

\[ y_1 = x, \quad y_1' = 1, \quad y_1'' = 0 \] so subs. get

\[ 0 - \frac{2}{x} \cdot 1 + \frac{2}{x^2} \cdot x = -\frac{2}{x} + \frac{2}{x} = 0 \quad \text{ok} \]

\[ y_2 = x^2, \quad y_2' = 2x, \quad y_2'' = 2 \] subs., get

\[ 2 - \frac{2}{x} \cdot 2x + \frac{2}{x^2} \cdot x^2 = 2 - 4 + 2 = 0 \quad \text{ok} \]

\[ W(x_1, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \]

\[ W(x_1, x^2) \neq 0 \text{ for } x > 0, \text{ so } x, x^2 \text{ are lin. indep solutions and the general solution to the DE is} \]

\[ y = c_1 x + c_2 x^2 \]