TOPICS:

- Springs and Masses
  - $\hookrightarrow$ Law + Assumptions
  - $\hookrightarrow$ Finding parameters
  - $\hookrightarrow$ Initial date times 2
- Simple Pendulum
  - $\hookrightarrow$ Linearize!
- Linear Equations... Again!

Springs and Masses

Hooke's Law:

$$\left[\text{ force exerted by a spring} = k \times \text{extension of the spring }\right]$$

Add Newton's Law and a coordinate system:

$$mx''(t) = -kx(t)$$

General Solution:

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

Where is Zero?

More Detail: Balance the Forces

- Force of Gravity: $F_{\mathrm{g}} = -mg$
- Spring Force: $F_{\mathrm{s}} = k \cdot (\mathrm{stretch}) = k \cdot \Big[(h - x(t)) - (h - x_{\mathrm{u}})\Big]$

$$mx''(t) = F_{\mathrm{total}} = F_{\mathrm{s}} + F_{\mathrm{g}}$$

or

$$mx''(t) = k \cdot \Big[x_u - x(t)\Big] - mg$$

or

$$mx''(t) = -kx(t) + \Big[k \cdot x_u - mg\Big]$$

Where is Zero?

You choose coordinates so that...

Example:

A 3kg mass stretches a spring 0.02 meters.

Suppose that a 2 kg mass is attached to the same spring and it is set in motion by pulling it down 5cm and releasing it from rest.

Write an initial value problem to model this situation.

Initial Data: <u>Position</u> and <u>Velocity</u>

A Little Qualitative Analysis

What does the model say?

$$mx''(t) = -kx(t)$$

- Increasing and Decreasing?
- Concave up and concave down?
- Steady States?

Characteristic Equations... Again

Look for solutions of the form $\sin(\omega t)$ and $\cos(\omega t)$:

$$m\Big(\sin(\omega t)\Big)'' + k\sin(\omega t) = 0$$

$$\big(-m\omega^2 + k\big)\sin(\omega t) = 0$$

Two roots:

$$\omega = \pm\sqrt{\frac{k}{m}}$$

Characteristic Equations... AGAIN!

Look for solutions of the form e^{rt} (just as for first-order).

Plug it in:

$$0 = m\Big(e^{rt}\Big)'' + \ ke^{rt} = \Big(mr^2 + k\Big)e^{rt}$$

Two roots:

$$r = \pm i\sqrt{\frac{k}{m}}$$

Add Friction Damping

<u>Assume</u>

- a friction force directed opposite the velocity vector
- magnitude of the force is proportional to the velocity

$$F_{\text{friction}} = -px'(t)$$

New Model:

$$mx''(t) + px'(t) + kx(t) = 0$$

Example

A force of 0.36 Newtons is required to drag a 7 kg mass across a table at a speed of 0.4 m/s.

$$0.36 = p \cdot (0.4) \Longrightarrow p = \frac{0.36}{0.4} = 0.9$$

Aside: Is this a coefficient of static or kinetic friction?

Simple Pendulum

Basic Trig:

If θ is measured in radians, then $L\theta$ is distance along the circumference of a circle.

$$\text{Acceleration} = (L\theta)'' = L\theta''$$

$\hookrightarrow$ Force of Gravity: $-mg\sin(\theta)$

$\hookrightarrow$ Model: $mL\theta'' = -mg\sin(\theta)$

Pictures: **at 11:00**

Linearize the Pendulum

Nonlinear Model:

$$L\theta'' = -g\sin(\theta)$$

"Recall"

$$\sin(\theta) = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots$$

$$L\theta'' = -g\sin(\theta) = g\theta - g\frac{1}{3!}\theta^3 + g\frac{1}{5!}\theta^5 + \cdots$$

When is the approximation good?

Linear Equations Again!

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

Just as for first-order:

General Solution = Homogeneous + Particular

Troubles?

- The homogeneous solution is more complicated.
- You cannot just separate and integrate.

  (V of P and UC will still work!)

Homogeneous Solutions

- $x'' + 2x = 0$ has two solutions:

  $x_1(t) = \sin(\sqrt{2}t)$ and $x_2(t) = \cos(\sqrt{2}t)$
- $y'' - 9y = 0$ has two solutions:

  $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$
- $y'' + 2y' + y = 0$ has two solutions:

  $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$

Superposition Still Works!

Linear Independence

Call two functions **linearly independent**

on an interval $a \le x \le b$ if

$$C_1\, y_1(x) + C_2\, y_2(x) = 0 \;\text{ on } a \le x \le b$$

implies that

$$C_1 = C_2 = 0$$

Key: This is a lot of equations in 2 unknowns!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—pick two values for t

$$\begin{aligned} 0 &= C_1 \sin(0) + C_2 \cos(0) \\ 0 &= C_1 \sin(\pi/2) + C_2 \cos(\pi/2) \end{aligned}$$

$\star$ This forces $C_1 = C_2 = 0$

There is another way!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—Differentiate the equation

$$\begin{aligned} 0 &= C_1 \sin(4t) + C_2 \cos(4t) \\ 0 &= 4C_1 \cos(4t) - 4C_2 \sin(4t) \end{aligned}$$

This has a nonzero solution if and only if

$$0 = \det \begin{bmatrix} \sin(4t) & \cos(4t) \\ 4\cos(4t) & -4\sin(4t) \end{bmatrix}$$

- ⋆ But this is *never* zero.
- ⋆ Name it: the <u>Wronskian</u>

HW #6: Due Friday in Lecture

Sec 3.3.4: 2, 5;

Sec 5.1.6: 1, 2, 4;

Friday in the Box: Section 6