TOPICS

- Some Basic Theory
  - $\hookrightarrow$ Linear Equations
  - $\hookrightarrow$ Linear Independence
- CE for Constant Coefficients
  - $\hookrightarrow$ Three Things Happen

General Theory: Linear Equations Again!

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

Just as for first-order:

General Solution = Homogeneous + Particular

Troubles?

- The homogeneous solution is more complicated.
- You cannot just separate and integrate.
  (V of P and UC will still work!)

Superposition Still Works!

If $y_1(x)$ and $y_2(x)$ solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

then $y_1(x) - y_2(x)$ solves...

Example: Homogeneous Solutions

$$x'' + 2x = 0$$

Two solutions:

$$x_1(t) = \sin(\sqrt{2}t) \quad \text{and} \quad x_2(t) = \cos(\sqrt{2}t)$$

Build some more...

Linear Independence

Call two functions **linearly independent**

on an interval $a \leq x \leq b$ if

$$C_1\, y_1(x) + C_2\, y_2(x) = 0 \quad \text{on } a \leq x \leq b$$

implies that

$$C_1 = C_2 = 0$$

Key: This is a lot of equations in 2 unknowns!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—pick two values for t

$$\begin{aligned} 0 &= C_1 \sin(0) + C_2 \cos(0) \\ 0 &= C_1 \sin(\pi/2) + C_2 \cos(\pi/2) \end{aligned}$$

$\star$ This forces $C_1 = C_2 = 0$

There is another way!

Example: $\sin(4t)$ and $\cos(4t)$ are L.I.D.

Solve a system—Differentiate the equation

$$\begin{aligned} 0 &= C_1 \sin(4t) + C_2 \cos(4t) \\ 0 &= 4C_1 \cos(4t) - 4C_2 \sin(4t) \end{aligned}$$

This has a nonzero solution if and only if

$$0 = \det \begin{bmatrix} \sin(4t) & \cos(4t) \\ 4\cos(4t) & -4\sin(4t) \end{bmatrix}$$

Breathe...

Theorem

Two solutions for

$$y'' + P(x)y' + Q(x)y = 0$$

are linearly independent on their domain if the Wronskian is not zero for at least one point in the interval.

Example:

Show that $y_1(x) = x^2$ and $y_2(x) = x^{-2}$ are linearly independent solutions for

$$x^2y'' + xy' - 4y = 0$$

Remarks:

- The Wronskian is a function of x.
- $W(x)$ satisfies its own ODE!

$$\frac{dW}{dx} + P(x)W(x) = 0$$

  if y_1 and y_2 are solutions for

$$y'' + P(x)y' + Q(x)y = 0$$

Summary: To solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1y_1(x) + C_2y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1y_1(x) + C_2y_2(x)$$

Characteristic Equations

The key to the constant coefficient universe!

To solve

$$mx'' + px' + kx = 0$$

assume that

$$x(t) = e^{rt}$$

and ... do what?

Characteristic Equation

$$mr^2 + pr + k = 0$$

Three Cases:

- Two, Distinct Real Roots: r_1 and r_2
- One, Repeated Real Root: $r = r_1 = r_2$
- Two, Complex Conjugate Roots:
$$r_1 = \alpha + i\beta \text{ and } r_2 = \alpha - i\beta$$

Distinct Roots $\Longrightarrow$ Life is Good

- $y'' - 9y = 0 \implies r^2 - 9 = 0$
$$r_1 = +3,\ r_2 = -3 \text{ so } y_1(x) = e^{3x} \text{ and } y_2(x) = e^{-3x}$$
- $x'' + 3x' + 2x = 0 \implies r^2 + 3r + 2 = 0$

  so... what?

Think backwards...

Find an equation with solutions

$$y_1(x) = e^{3x} \quad y_2(x) = e^{-(1/2)x}$$

Repeated Root: Life is Pretty Good

- $y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = 0$

  $r_1 = -2$ and $r_2 = -2$ gives

  $y_1(x) = e^{-2x}$ and $y_2(x) = ???$

To build another solution, assume that

$$y_2(x) = u(x)y_1(x)$$

and do what?

Plug it in

$$0 = \Big(u(x)y_1\Big)'' + 4\Big(u(x)y_1\Big)' + 4\Big(u(x)y_1\Big)$$

The whole thing reduces to

$$u''y_1(x) = 0$$

Test Independence

$$y_1(x) = e^{rx} \quad \text{and} \quad y_2(x) = xe^{rx} \quad \text{are L.I.D.}$$

Compute the Wronskian:

$$W(y_1, y_2) = \det \begin{bmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx} + rxe^{rx} \end{bmatrix}$$

You get

$$W(y_1, y_2) = e^{2rx} + rxe^{2rx} - rxe^{2rx} = e^{2rx}$$

$\star$ and this is *never* zero $\star$

Complex Conjugate Roots ....Next Time?

HW #8: Due Tuesday in Conference

Sec 5.2.2: 4, 5, 10 ;

Sec 6.1.5: 12, 13, 15 ;

Sec 6.2.1: 1, 5 ;