TOPICS

- Characteristic Equations
  - $\hookrightarrow$ Complex Conjugate Case
- Undetermined Coefficients
  - $\hookrightarrow$ Constant Coefficients + Special Forcing Terms
- Numerical Methods
  - $\hookrightarrow$ Euler is Back!

Summary: To solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1y_1(x) + C_2y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1y_1(x) + C_2y_2(x)$$

Complex Example... Olde Newes

Standard, unforced, undamped spring model:

$$mx'' + kx = 0 \implies mr^2 + k = 0$$

$$r_1 = +i\sqrt{\frac{k}{m}} \quad \text{and} \quad r_2 = -i\sqrt{\frac{k}{m}}$$

Solutions???

$$x_1(t) = \sin\left(\sqrt{\frac{k}{m}}t\right), \quad x_2(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$$

Euler's Formula:

Connect complex exponentials and trig functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

And the other direction...

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Complex Example

Solve

$$y'' + 2y' + 2y = 0$$

CE: $r^2 + 2r + 2 = 0$ has roots

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \frac{1}{2}\sqrt{-4} = -1 \pm i$$

Continue the Example:

You have L.I.D. solutions:

$$y_1(x) = e^{(-1+i)x} \quad \text{and} \quad y_2(x) = e^{(-1-i)x}$$

A more useful form:

$$e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}\Big[\cos(x) + i\sin(x)\Big]$$

$$e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}\Big[\cos(x) - i\sin(x)\Big]$$

Continue the Example:

Compare Two General Solutions

$$y(x) = c_1 e^{(-1+i)x} + c_2 e^{(-1-i)x}$$

and

$$y(x) = e^{-x}\left[A\sin(x) + B\cos(x)\right]$$

Questions:

Are these the same solution?

How are the constants related?

Example: Solve an IVP

$$y''(x) + 2y'(x) + 2y(x) = 0, \quad y(0) = 10,\ y'(0) = 5$$

Solve for A and B

$$\begin{aligned} 10 = y(0) &= e^0 \Big[A\sin(0) + B\cos(0) \Big] = B \\ 5 = y'(0) &= \end{aligned}$$

So: $A =$ __________ and $B =$ ______________

Remarks:

- A and B are Real
- If you use the exponentials, then...

  c_1 and c_2 are *complex*

$$c_1 = 5 + \frac{15}{2}i, \quad c_2 = 5 - \frac{15}{2}i$$

Add a Forcing Term

Solve

$$x''(t) + 4x(t) = \sin(t)$$

- Step 1: $x_h(t) = c_1 \sin(2t) + c_2 \cos(2t)$

  $\hookrightarrow$ **C.E.** $r^2 + 4 = 0 \implies r = \pm 2i$

- Step 2: Find a particular solution:

Dirty Work:

Try $x_p(t) = A\sin(t) + B\cos(t)$

Plug it in an solve for A and B.

Summary:

General Solution:

$$x(t) = \frac{1}{3}\sin(t) + C_1 \sin(2t) + C_2 \cos(2t)$$

Add some initial data...

Undetermined Coefficients Works... When

- Constant Coefficients
- Forcing Term has a finite set of linearly independent derivatives.
- $f(x) = e^{\lambda x}$ .... $y_p(x) = Ae^{\lambda x}$
- $f(x) = \sin(\omega t)$ .... $y_p(x) = A\sin(\omega t) + B\cos(\omega t)$
- $f(x) = x^n$ ..... $y_p(x) = A_0 + A_1 x^1 + A_2 x^2 + \cdots + A_n x^n$

Remarks:

- ⋆ It works for <u>products</u> too!:

$$f(x) = e^{2x}\sin(5x) \implies y_p(x) = e^{2x}\Big[A\sin(5x) + B\cos(5x)\Big]$$

- ⋆ Watch for homogeneous solutions!

Example: Exponentials All Around

Solve $y'' + 4y' + 4y = 5e^{-2x}$

- CE gives $r_1 = -2$ and $r_2 = -2$ so

  $$y_h(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

- UC says to look for $y_p(x) = ???$

  $y_p(x) = Ae^{-2x}$ won't work.

The Dirty Work:

Guess $y_p(x) = Ax^2e^{-2x}$ and ____ __ __

Summary:

The general solution for $y'' + 4y' + 4y = 5e^{-2x}$

$$y(x) = C_1e^{-2x} + C_2xe^{-2x} + \frac{5}{2}x^2e^{-2x}$$

Questions:

- Can you change the initial data so that the final solution contains only the particular solution?
- Does the solution have a transient component?
- Can you change the initial data so that the final solution contains only the homogeneous solution?

Minute Quiz:

Find general solution for $y'' + 4y' + 4y = \left(5 \times 10^{23}\right) e^{-2x}$

Answer

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \left[10^{23} \times\right] \frac{5}{2} x^2 e^{-2x}$$

Euler's Method for Second-Order Equations

To approximate the solution to

$$mx''(t) + px'(t) + kx(t) = f(t), \quad x(0) = x_i,\ x'(0) = v_i$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= \frac{1}{m} f(t) - \frac{p}{m} v(t) - \frac{k}{m} x(t) \end{aligned}$$

Use Euler as before:

$$\begin{aligned} x(t+h) &= x(t) + v(t) \cdot h \\ v(t+h) &= v(t) + \left[\frac{1}{m} f(t) - \frac{p}{m} v(t) - \frac{k}{m} x(t)\right] \cdot h \end{aligned}$$

Example:

To approximate the solution to

$$x''(t) + 4x(t) = 0, \quad x(0) = 1,\ x'(0) = 0$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t), \quad x(0) = 1 \\ v'(t) &= -4x(t), \quad v(0) = 0 \end{aligned}$$

And Approximate:

$$\begin{aligned} x(t+h) &= x(t) + v(t) \cdot h, \quad x(0) = 1 \\ v(t+h) &= v(t) - 4x(t) \cdot h, \quad v(0) = 0 \end{aligned}$$

Take a Few Steps:

$$\begin{aligned} x_{n+1} &= x_n + v_n \cdot h, \quad x_0 = 1 \\ v_{n+1} &= v_n - 4x_n \cdot h, \quad v_0 = 0 \end{aligned}$$

$$\left[\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline t & 0 & 0.10 & 0.20 & 0.30 & 0.40 & 0.50 \\ x & 1 & 1.00 & 0.96 & 0.88 & 0.76 & 0.68 \\ v & 0 & -0.40 & -0.80 & -1.18 & -1.54 & -1.84 \end{array}\right]$$

HW #8: Due Tuesday in Conference

Sec 5.2.2: 4, 5, 10 ;

Sec 6.1.5: 12, 13, 15 ;

Sec 6.2.1: 1, 5 ;

HW #9: Due Friday in Lecture

Sec 6.3.4: 3, 4, 5, 21;

Sec 6.4.4: 7, 8, 17, 18 ;

Sec 6.5.4: 1, 2, 4 ;