TOPICS:

- Questions about Springs and Masses
  - ↪ How much damping is enough?
  - ↪ What is natural frequency?
  - ↪ How is amplitude related to initial data?
  - ↪ How is amplitude related to the forcing term?

Damping the Springs

$$mx'' + px' + kx = 0$$

Factoids:

(1) If $p > 0$ then the solutions always decay to zero.

(2) If p is *small*, then the solution oscillates for a long time before it dies out.

(3) If p is *large*, then the solution does not oscillate at all.

WHERE does the behavior change?

C.E. Tells All:

$$mx'' + px' + kx = 0 \quad \Longrightarrow \quad mr^2 + pr + k = 0$$

Look at the roots: $$r = \frac{-p \pm \sqrt{p^2 - 4mk}}{2m}$$

(i) If $p^2 - 4mk > 0$, then...

(ii) If $p^2 = 4mk$, then...

(iii) If $p^2 < 4mk$ then...

Examples:

$$x'' + px' + x = 0, \quad x(0) = 2,\ x'(0) = 0$$

- $p^2 < 4 \longrightarrow$ underdamping.
- $p^2 = 4 \longrightarrow$ critical damping.
- $p^2 > 4 \longrightarrow$ overdamping.

Solutions:

$$x_1(t) = \mathrm{e}^{\left(-1/2\,\left(p-\sqrt{p^2-4}\right)\,t\right)}$$

More of the Example:

$$x'' + px' + x = 0, \quad x(0) = 2,\ x'(0) = 0$$

Solution:

$$\begin{aligned} x(t) &= \frac{\left(p^2 - 4 + \sqrt{p^2 - 4}\,p\right)}{p^2 - 4}\, e^{\left(-1/2\left(p - \sqrt{p^2-4}\right)t\right)} \\ &\quad + \frac{\left(p^2 - 4 - \sqrt{p^2 - 4}\,p\right)}{p^2 - 4}\, e^{\left(-1/2\left(p + \sqrt{p^2-4}\right)t\right)} \end{aligned}$$

What happens when $p = 2$?

Amplitude and Initial Data

$$x''(t) + 4x(t) = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

General Solution:

$$x(t) = C_1 \sin(2t) + C_2 \cos(2t)$$

Solve for the constants:

$$\begin{aligned} x_i = x(0) &= C_1 \cdot 0 + C_2 \cdot 1 \\ v_i = x'(0) &= 2C_1 \cdot 1 - 2C_2 \cdot 0 \end{aligned}$$

Conclude:

A Little Calculus

What is the maximum value for $x(t) = C_1 \sin(2t) + C_2 \cos(2t)$?

$$\frac{dx}{dt} = 2C_1 \sin(2t) - 2C_2 \cos(2t) = 0 \iff \frac{\sin(2t)}{\cos(2t)} = \frac{C_1}{C_2}$$

"Critical Point" $\qquad 2t^* = \arctan\left(\frac{C_1}{C_2}\right)$

For this value t^*,

$$x(t^*) = C_1 \sin\left(\arctan\left(\frac{C_1}{C_2}\right)\right) + C_2 \cos\left(\arctan\left(\frac{C_1}{C_2}\right)\right)$$

A Little Trig

Conclude the Example:

$$x''(t) + 4x(t) = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

The solution has

$$\text{Amplitude} \;\; = \sqrt{\left(x_i\right)^2 + \left(\frac{v_i}{2}\right)^2}$$

Aside:

Write the general solution as

$$x(t) = A \sin\Big(\omega_n t + \phi\Big)$$

where

$$A = \sqrt{\;x_i^2 + \left(\frac{v_i}{\omega_n}\right)^2}$$

and $\phi =$???

- This is the Phase Angle–Amplitude form of the solution.
- What is the period?

Amplitude and Forcing

Question: How does the amplitude of the solution depend on the frequency of the forcing term?

Answer:

Amplitude and Forcing

Example:

$$x'' + 4x = \cos(\omega t), \quad x(0) = 0,\ x'(0) = 0$$

Solve it:

$$x(t) = \frac{-1}{4-\omega^2}\cos(2t) + \frac{1}{4-\omega^2}\cos(\omega t)$$

Key: The amplitude becomes LARGE when $\omega \approx 2$

Key: When $\omega = 2$, this is NOT the solution.

Aside: Visit Maple

What Happens when $\omega = \omega_n$?

Solve

$$x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = x_i,\ x'(0) = v_i$$

The particular solution will be

$$x_p(t) = At\sin(\omega t) + Bt\cos(\omega t)$$

Plug it in...

The Dirty Work:

$$\begin{aligned} F_0 \sin(\omega t) &= \Big[2\omega A\cos(\omega t) - \omega^2 At\sin(\omega t) \\ &\qquad - 2\omega B\sin(\omega t) - \omega^2 Bt\cos(\omega t)\Big] \\ &\quad + \omega^2\Big[At\sin(\omega t) + Bt\cos(\omega t)\Big] \\ &= \Big[2\omega A\cos(\omega t) - 2\omega B\sin(\omega t)\Big] \end{aligned}$$

$$x_p(t) = -\frac{F_0}{2\omega} t\cos(\omega t)$$

Minute Quiz:

How long will it take the solution

$$x'' + 4x = 10\sin(2t), \quad x(0) = 0,\ x'(0) = 0$$

to reach +100?

Approximate answer:

$$x_p(t) = -\frac{10}{4} t\cos(2t) \geq 100$$

HW #9: Due Friday in Lecture

Sec 6.3.4: 3, 4, 5, 21;

Sec 6.4.4: 7, 8, 17, 18 ;

Sec 6.5.4: 1, 2, 4 ;

Project #2:

Epidemics OR **Playgrounds**