TOPICS:

- Basic Models

  $\hookrightarrow$ How can you identify parameters?

- Resonance for Damped Systems

  $\hookrightarrow$ What is the maximum amplitude?

- Life in the Phase Plane

  $\hookrightarrow$ Steady States and Limiting Solutions

Example:

Attach a mass $m = 2$ to a spring with $k = 4$ and set it in motion from the equilibrium position with some (unknown) initial velocity v_i.

- If the measured amplitude of the motion is $12cm$, what is the initial velocity?

- If the period is measured to be 0.9 seconds, what is the initial velocity?

Example Extended:

$$2x'' + px' + 4x = 0, \quad x(0) = x_i,\ v(0) = 0$$

The amplitude decays from $12cm$ to $8cm$ after 2 minutes, can you determine the damping coefficient?

Long-term Behavior

$$mx'' + px' + kx = F_0 \sin(\omega t)$$

(1) If $p > 0$, the homogeneous solution is *transient*...

(2) Find all steady states...

(3) Describe limiting behavior...

Homogeneous Solutions are Transient:

The Roots are negative: $r = \dfrac{-p \pm \sqrt{p^2 - 4mk}}{2m} < 0$

For the complex case:

$$x_h(t) = e^{-\frac{p}{2m}t}\Big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\Big]$$

where $\beta = \dfrac{1}{2m}\sqrt{4mk - p^2}$

Evaluate the limit:

$$\lim_{t\to\infty} x_h(t) = ???$$

Dirty Work

Example:

$$x'' + \frac{1}{4}x' + x = 2\cos(3t), \quad x(0) = 2,\ x'(0) = 0$$

Solution:

$$\begin{aligned} x(t) &= e^{-\frac{1}{8}t}\left[C_1 \sin\left(\frac{3}{8}\sqrt{7}\,t\right) + C_2 \cos\left(\frac{3}{8}\sqrt{7}\,t\right)\right] \\ &\quad + \frac{24}{1033}\sin(3t) - \frac{256}{1033}\cos(3t) \end{aligned}$$

When is the transient part smaller than 0.0001?

Limiting Periodic Solution

$$x'' + \frac{1}{4}x' + x = 2\cos(3t), \quad x(0) = 2,\ x'(0) = 0$$

- The equation has no steady state.
- All solutions converge to the same solution.

  Look at pictures...

Look at the Phase Plane...

Graph

$$\begin{aligned} x(t) &= C_1 \sin(2t) + C_2 \cos(2t) \\ v(t) &= 2C_1 \cos(2t) - 2C_2 \sin(2t) \end{aligned}$$

Look at

$$\left(x(t)\right)^2 + \left(\frac{v(t)}{2}\right)^2$$

$$\begin{aligned} &= C_1^2 \sin^2(2t) + 2C_1C_2 \sin(2t)\cos(2t) + C_2^2 \cos^2(2t) \\ &\quad + C_1^2 \sin^2(2t) - 2C_1C_2 \sin(2t)\cos(2t) + C_2^2 \cos^2(2t) \\ &= \ldots \end{aligned}$$

Wait a Minute...

Define the Total Energy:

$$E(t) = \frac{1}{2} m v^2(t) + \frac{1}{2} k x^2(t)$$

Differentiate it:

$$\begin{aligned} \frac{d}{dt} E(t) &= \\ &= \\ &= \\ &= -p\left(x'(t)\right)^2 \end{aligned}$$

Implications:

- If $p = 0$, then $E(t)$ is constant.
- If $p = 0$, the solution are ellipses in the phase plane:

$$\frac{1}{2}mv^2(t) + \frac{1}{2}kx^2(t) = E(0)$$

- If $p > 0$, then $E(t)$ is decreasing.

  (The graphs in the phase plane are spirals...)

Pictures in the Phase Plane:

Example:

$$x'' + px' + x = 0, \quad x(0) = x_i,\ x'(0) = v_i$$

Turn it into a system

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -x(t) - pv(t) \end{aligned}$$

Look at solutions for different values of p.

Note: The Maple worksheets that generated all of the graphs in this lecture are available on the course homepage.

Nullclines:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -x(t) - pv(t) \end{aligned}$$

- $x'(t) = 0$ when ________________
- $v'(t) = 0$ when ________________

Pictures:

Euler's Method for Second-Order Equations

To approximate the solution to

$$mx''(t) + px'(t) + kx(t) = f(t), \quad x(0) = x_i,\ x'(0) = v_i$$

⋆ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= \frac{1}{m}f(t) - \frac{p}{m}v(t) - \frac{k}{m}x(t) \end{aligned}$$

Use Euler as before:

New Value = Old Value + (Slope)×(time step)

Example:

To approximate the solution to

$$x'' + 2x' + 4x = 0, \quad x(0) = 1,\ x'(0) = 0$$

⋆ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t), & x(0) = 1 \\ v'(t) &= -2v(t) - 4x(t), & v(0) = 0 \end{aligned}$$

And Approximate:

$$\begin{aligned} x(t+h) &= x(t) + v(t)\cdot h, & x(0) = 1 \\ v(t+h) &= v(t) - (2v(t) + 4x(t))\cdot h, & v(0) = 0 \end{aligned}$$

Take a Few Steps:

$$\begin{aligned} x_{n+1} &= x_n + v_n \cdot h, \\ v_{n+1} &= v_n - (2v_n + 4x_n) \cdot h \end{aligned}$$

$$\left[\begin{array}{c|cccccc} n & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline t & 0 & 0.50 & 1.00 & 1.50 & 2.00 & 2.50 \\ x & 1 & 1.00 & & & & \\ v & 0 & -2.00 & & & & \end{array}\right]$$

HW #9: Due Thursday

Sec 7.1.1: 3, 4, ;

Sec 7.2.4: 1, 3, 4 ;

Sec 8.1.4: 1(d,e,g), 7 ;

Bonus Problem: Due Wednesday

Most engineers would take the general spring-mass model

$$mx'' + px' + kx = F\sin(\omega_f t)$$

and rewrite it as

$$x'' + 2\zeta x' + \omega^2 x = F_0 \sin(\omega_f t)$$

Explain how they do this. What values of ζ admit a resonance phenomenon in the general model? What is the resonant frequency?