TOPICS

- Coupled Systems:
  - $\hookrightarrow$ Predator-Prey Systems
  - $\hookrightarrow$ Competing Species
- Systems of Equations
  - $\hookrightarrow$ Stability in Competition
  - $\hookrightarrow$ General Case

Systems of Differential Equations

Basic Predator-Prey Model

$$\begin{aligned} F' &= -(d_F - \alpha R)F \\ R' &= +(b_R - \beta F)R \end{aligned}$$

Qualitative Analysis

- Steady State Population:

$$\begin{aligned} F' = 0 &= -(d_F - \alpha R)F \\ R' = 0 &= +(b_R - \beta F)R \end{aligned}$$

- Two solutions:

$$(F, R) = (0, 0), \quad (F, R) = \left(\frac{b_R}{\beta}, \frac{d_F}{\alpha}\right)$$

More Qualitative Analysis

- $F' > 0 \Longleftarrow$
- $R' > 0 \Longleftarrow$

Sketch a direction field…

Pictures...

Competing Species

$$
\begin{aligned}
x'(t) &= x\Big(3 - x - y\Big) \\
y'(t) &= y\left(2 - y - \frac{1}{2}x\right)
\end{aligned}
$$

Find the Steady States:

..................................Which of these are STABLE?

Visit Maple...

Linearize Around a Steady State

Define "perturbations" from $(x_{ss}, y_{ss}) = (2, 1)$

$$\begin{aligned} x(t) &= 2 + p_1(t) \\ y(t) &= 1 + p_2(t) \end{aligned}$$

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$$\begin{aligned} p_1' &= (p_1 + 2)\Big(3 - (p_1 + 2) - (p_2 + 1)\Big) \\ &= \end{aligned}$$

$$\begin{aligned} p_2' &= (p_2 + 1)\Big(2 - (p_2 + 1) - \frac{1}{2}(p_1 + 2)\Big) \\ &= \end{aligned}$$

Linearized Equation:

$$
\begin{aligned}
p_1' &= -2p_1 - 2p_2 \\
p_2' &= -\frac{1}{2}p_1 - p_2
\end{aligned}
$$

Question: How do you solve this system for p_1 and p_2?

Answer:

Characteristic Equation!

Look for a <u>solution pair</u>

$$(p_1(t), p_2(t)) = \left(ae^{rt}, be^{rt}\right)$$

and solve for a, b, and r.

What is he talking about?

Plug it in: Derive C.E.

$$\begin{aligned} rae^{rt} &= -2ae^{rt} - 2be^{rt} \\ rbe^{rt} &= -\frac{1}{2}ae^{rt} - be^{rt} \end{aligned}$$

$$\begin{array}{rcrcl} (r+2)a & + & 2b & = & 0 \\ \frac{1}{2}a & + & (r+1)b & = & 0 \end{array}$$

Easy solution: $a = b = 0 \ldots$ but that is useless!

A Little Algebra...

Characteristic Equation:

$$(r+2)(r+1)-1=0 \implies r^2+3r+1=0$$

So ...

$$r_1 = \underline{\hspace{3cm}}, r_2 = \underline{\hspace{3cm}}$$

You still have to find a and b

The Story So Far...

- The steady state $(2, 1)$ is stable.
- Find non-trivial solutions for special exponents r_1 and r_2.
- You still need to find a and b.

Aside:

For $r_1 = -\frac{3}{2} + \frac{\sqrt{5}}{2}$:

$$\begin{array}{rcrcl} (r_1+2)a & - & 2b & = & 0 \\ \frac{1}{2}a & + & (r_1+1)b & = & 0 \end{array}$$

General Case:

$$\begin{array}{ccccc} y' & = & ay & + & bz \\ z' & = & cy & + & dz \end{array}$$

$$\begin{array}{ccccc} (r-a)p & - & bq & = & 0 \\ -cp & + & (r-d)q & = & 0 \end{array}$$

CE:

$$\det \begin{bmatrix} (r-a) & -b \\ -c & (r-d) \end{bmatrix} = 0$$

General Solution:

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} e^{r_1 t} + \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} e^{r_2 t}$$

Welcome to Matrix Notation

Reality Check:

$$x'' + px' + kx = 0$$

Turn it into a system

$$\begin{array}{rcrcl} x'(t) & = & 0x(t) & + & 1v(t) \\ v'(t) & = & -kx(t) & - & pv(t) \end{array}$$

New Characteristic Equation:

$$\det \begin{bmatrix} (r-0) & -1 \\ k & (r+p) \end{bmatrix} = r(r+p) + k = r^2 + pr + k = 0$$

HW #9: Due Thursday

Sec 7.1.1: 3, 4, ;

Sec 7.2.4: 1, 3, 4 ;

Sec 8.1.4: 1(d,e,g), 7 ;