TOPICS

- Systems of Equations
  - $\hookrightarrow$ A Little Theory
- When Coefficients are Constant...
  - $\hookrightarrow$ More Characteristic Equations
- Linear Algebra Approach
  - $\hookrightarrow$ Eigenthingies

Announcements:

- **HW #10: Due Monday in Lecture**

  Sec 8.2.6: 1, 3, 13, 15, 16 ;

  Sec 8.3.5: 1, 3, 4, 17 ;
- Test II:
  - $\hookrightarrow$ Next Wednesday in Lecture
  - $\hookrightarrow$ Re-Test Thursday at 4:30pm in SL115

Tests will be graded by 4:00pm on Wednesday

No Conference on Thursday

Last Time...

$$\begin{aligned} y' &= ay + bz \\ z' &= cy + dz \end{aligned}$$

$$\begin{aligned} (r-a)p \;-\; \quad bq &= 0 \\ -cp + (r-d)q &= 0 \end{aligned}$$

CE:

$$\det \begin{bmatrix} (r-a) & -b \\ -c & (r-d) \end{bmatrix} = 0$$

General Solution:

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} e^{r_1 t} + \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} e^{r_2 t}$$

Welcome to Matrix Notation

System of ODEs

$$y' = f(t,y,z), \quad y(t_0) = y_0$$

$$z' = g(t,y,z), \quad z(t_0) = z_0$$

LINEAR System of ODEs

$$\begin{aligned} y' &= f_1(t)y + f_2(t)z \\ z' &= g_1(t)y + g_2(t)z \end{aligned}$$

Note: The system is homogeneous. . . because?

For LINEAR Systems:

- General Solution requires . . .
  - $\hookrightarrow$ two linearly independent solution pairs
  - $\hookrightarrow$ a particular solution-n
- Today's equations: Homogeneous

Example: All of the Details

$$\begin{array}{rcrcr} y' & = & -y & + & 3z \\ z' & = & 3y & - & z \end{array}$$

Look for a solution pair

$$(y(t), z(t)) = \left(pe^{rt}, qe^{rt}\right)$$

and solve for p, q, and r.

Plug it in: Derive C.E.

$$\begin{array}{rcl} rpe^{rt} & = & -pe^{rt} + 3qe^{rt} \\ rqe^{rt} & = & 3pe^{rt} - qe^{rt} \end{array}$$

$$\begin{array}{rcrcl} (r+1)p & - & 3q & = & 0 \\ -3p & + & (r+1)q & = & 0 \end{array}$$

CE:

$$\det \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \qquad\qquad = 0$$

Distinct, Real Roots

$$(r+1)(r+1) - 9 = 0 \implies r_1 = +2,\ r_2 = -4$$

You still have to find p and q

<u>For $r_1 = +2$:</u>

$$\begin{array}{rcrcl} (2+1)p & - & 3q & = & 0 \\ -3p & + & (2+1)q & = & 0 \end{array}$$

Second Solution:

<u>For $r_2 = -4$:</u>

$$\begin{array}{rcrcl} (-4+1)p & - & 3q & = & 0 \\ -3p & + & (-4+1)q & = & 0 \end{array}$$

Both equations require ...

Conclusion: you have the general solution

$$\begin{aligned}\Big(y(t), z(t)\Big) &= C_1\Big(y_1(t), z_1(t)\Big) + C_2\Big(y_2(t), z_2(t)\Big)\\ &= \Big(C_1e^{2t} + C_2e^{-4t}, C_1e^{2t} - C_2e^{-4t}\Big)\end{aligned}$$

Loose Ends:

Are the solution pairs linearly independent?

The Wronskian works again:

$$W(t) = \det\begin{bmatrix} y_1(t) & y_2(t) \\ z_1(t) & z_2(t)\end{bmatrix} = y_1(t)z_2(t) - y_2(t)z_1(t)$$

If this is not zero for at least one t, the solution pairs are linearly independent and you really do have the general solution.

For this example,

$$W(t) = (e^{2t})(-e^{-4t}) - (e^{-4t})(e^{2t}) = -2e^{-2t}$$

Aside:

Once you have $y_1(t)$ and $y_2(t)$, use the ODE to find the z's

$$\text{ODE} \quad \left\{ \begin{array}{rcrcr} y' & = & -y & + & 3z \\ z' & = & 3y & - & z \end{array} \right.$$

$$z(t) = \frac{1}{3}\Big(y'(t) + y(t)\Big)$$

$$y_1(t) = e^{2t} \implies$$

$$y_2(t) = e^{-4t} \implies$$

Matrix View:

$$\begin{array}{rcrcr} y' & = & ay & + & bz \\ z' & = & cy & + & dz \end{array}$$

$$\begin{array}{rcrcr} (r-a)p & - & bq & = & 0 \\ -cp & + & (r-d)q & = & 0 \end{array}$$

CE:

$$\det \begin{bmatrix} (r-a) & -b \\ -c & (r-d) \end{bmatrix} = 0$$

Matrix View Again:

System:

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}' = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$

Guess

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} e^{rt}$$

and plug it in...

Aside: Multiplying Matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} ay + bz \\ cy + dz \end{bmatrix}$$

And Again:

$$r \begin{bmatrix} p \\ q \end{bmatrix} e^{rt} = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} e^{rt}$$

You get an equation for r, p, and q:

$$\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} p \\ q \end{bmatrix} = r \begin{bmatrix} p \\ q \end{bmatrix}$$

In matrix notation:

$$A\vec{p} = r\vec{p}$$

Definition:

Given a square matrix A, an *eigenvector* is a non-trivial vector $\vec{v}$ such that

$$A \cdot \vec{v} = r \cdot \vec{v}$$

for some constant r.

An amazing thing...

A linear system can be written

$$\frac{d}{dt}\vec{y} = A \cdot \vec{y}$$

where A is a square matrix.

The solution is

$$\vec{y}(t) = \vec{y_0} \cdot e^{At}$$

(as soon as you figure out how to exponentiate a matrix!)

Project #2: Review in Conference Tomorrow