TOPICS

- Population Models

$\hookrightarrow$ One Approach: Analyze Some Data

$\hookrightarrow$ Another Approach: Balance Laws

$\hookrightarrow$ Examples: Emigration, Harvesting, Limited Growth

- HW #2: Due Thursday

Solve an IVP

$$P'(t) = kP(t), \quad P(t_0) = P_0$$

Separate:

$$\frac{dP}{P} = k \cdot dt$$

Integrate:

$$\ln(P(t)) = k\ dt + c \Longrightarrow P(t) = Ce^{kt}$$

What is C?

Find the Constant

$$P_0 = P(t_0) = Ce^{k \cdot t_0}$$

So

$$C = \frac{p_0}{e^{k \cdot t_0}}$$

and

$$P(t) = P_0 \; e^{k(t - t_0)}$$

DATA: U.S. Population

Year	$P(t)$	$D(t)$	$D(t)/P(t)$
1790	3.93	$\cdots$	$\cdots$
1800	5.30	0.164	$\cdots$
1810	7.22	$\cdots$	$\cdots$
1820	9.62	$\cdots$	$\cdots$
1830	12.90	$\cdots$	$\cdots$
1840	17.12	$\cdots$	$\cdots$
1850	23.26	$\cdots$	$\cdots$

U.S. Population

Year	$P(t)$	$D(t)$	$D(t)/P(t)$
1790	3.93	$\cdots$	$\cdots$
1800	5.30	0.164	0.031
1810	7.22	0.216	0.030
1820	9.62	0.284	0.030
1830	12.90	0.375	0.029
1840	17.12	0.518	0.030
1850	23.26	$\cdots$	$\cdots$

DERIVE the Model

$$D(t) = \frac{P(t+10) - P(t-10)}{20} \quad \text{and} \quad D(t) \approx \frac{dP}{dt}$$

Model:

$$\boxed{\frac{1}{P(t)} \cdot \frac{dP}{dt} = 0.030}$$

Analyze the Model

$$\frac{dP}{dt} = 0.03P(t)$$

General Solution:

$$P(t) = Ce^{(0.03 \cdot t)}$$

Use the Data from 1790 to get the Particular Solution:

$$P(t) = 3.93\ e^{0.03(t-1790)}$$

Year	Actual	Model
1790	3.93	3.93
1800	5.30	5.30
1810	7.22	7.16
1820	9.62	9.66
1830	12.90	13.05
1840	17.12	17.61
1850	23.26	23.77

Model: P(2000) = 2140.2 (million people)

Exponential Growth

Work with Excel...

Alternate View: Balance Equation

Net Change in Δt = (rate in)$\times\Delta t$ + (rate out)$\times\Delta t$

$\hookrightarrow$ Net Change $= P(t + \Delta t) - P(t)$

$\hookrightarrow$ Rate In $= k_{\text{in}} P(t)$

$\hookrightarrow$ Rate Out $= k_{\text{out}} P(t)$

Transition to ODE

$$[P(t+\Delta t) - P(t)] = k_{\text{in}} P(t) \Delta t - k_{\text{out}} P(t) \Delta t$$

$$\frac{[P(t+\Delta t) - P(t)]}{\Delta t} = (k_{\text{in}} - k_{\text{out}})\, P(t)$$

Take the limit as $\Delta t \to 0$ to get an ODE:

$$\frac{dP}{dt} = kP(t)$$

where $k = k_{\text{in}} - k_{\text{out}}$

Start Harvesting

Same Balance Equation (almost):

$$P(t+\Delta t) - P(t) = [k_{\text{in}} P(t) - k_{\text{out}} P(t) - H]\, \Delta t$$

Divide by Δt and let $\Delta t \to 0$:

$$\frac{dP}{dt} = kP(t) - H$$

Very different picture

Logistic Growth

Balance Equation:

$$[P(t+\Delta t) - P(t)] = [aP(t)]\,\Delta t - \left[s(P(t))^2\right]\Delta t$$

$$\text{Death Rate} = sP^2 \Longleftarrow \text{Competition}$$

ODE: Take a Limit

$$\frac{dP}{dt} = aP(t) - s(P(t))^2 = (a - sP(t)) \cdot P(t)$$

Look at More Data:

Decreasing Growth Rate?

Sketch Some Solutions

$$\frac{dP}{dt} = (a - sP) \cdot P, \quad P(t_0) = P_i$$

$\surd$ $\quad \dfrac{dP}{dt} = 0$ when $P = 0$ or $a - sP = 0$

$\surd$ $\quad \dfrac{dP}{dt} > 0$ when $a - sP > 0$, or $P < a/s$

$\surd$ $\quad \dfrac{dP}{dt} < 0$ when $a - sP < 0$, or $P > a/s$

a/s is the *Carrying Capacity*

HW #2: Due Thursday in Conference

$\hookrightarrow$ **Sec 2.1.7: 3d, 7, 13a, 13c**;

$\hookrightarrow$ **Sec 2.2.4: 2(a,c,f), 9**;

$\hookrightarrow$ **Section C01 in the box on Wednesday**