TOPICS

- You've ALMOST made it!
  - $\hookrightarrow$ TEST II: Wednesday at 9:00am
  - $\hookrightarrow$ Closed-book, No notes, No Calculator
- REVIEW PROBLEMS

Announcements:

- Review Session for Test II:

  $\hookrightarrow$ Tuesday at 7:00pm in SL115

Guaranteed Problems:

1. Solve a second-order equation
   - ↪ Homogeneous Solution & Particular Solution
   - ↪ Initial Data
2. Analyze a spring-mass system
   - ↪ Period, Frequency, Amplitude...
   - ↪ Under, Over, Critical Damping...
   - ↪ Resonance...
3. Solve a system of linear equations
   - ↪ CE and Homogeneous Solution
   - ↪ Eigenvalues and Eigenvectors
   - ↪ Plots in the Phase Plane

Problem #1: Analyze a Linear Equation

$$y'' + 3y' + 2y = 3\cos(2t) - \sin(2t)$$

(a) Solve the homogeneous equation.

(b) Find the particular and the general solution.

(c) Add initial data.

(d) Find the transient component.

(e) Is there a limiting periodic solution?

(a) Homogeneous Solution: CE

$$y'' + 3y' + 2y = 0$$

Characteristic Equation:

Roots $\Longrightarrow$ Two Solutions

Test Linear Independence: $W(y_1, y_2) = ???$

(b) Undetermined Coefficients:

$$y_p(t) = A\sin(2t) + B\cos(2t)$$

Plug it in:

$$\begin{aligned} 3\cos(2t) - \sin(2t) &= y_p'' + 3y_p' + 2y_p \\ &= (-4A - 6B + 2A)\sin(2t) \\ &\quad +(-4B + 6A + 2B)\cos(2t) \end{aligned}$$

Sine Term: $-2A - 6B = -1$

Cosine Term: $6A - 2B = 3$

$$A = \frac{1}{2},\ B = 0 \implies y_p(t) = \frac{1}{2}\sin(2t)$$

(b) & (c) Keep Going...

- General Solution: $y(t) = C_1 e^{-t} + C_2 e^{-2t} + \frac{1}{2}\sin(2t)$
- Find the <u>forced solution</u>: Set $y(0) = 0$, $y'(0) = 0$.

$$\begin{aligned} 0 &= y(0) = C_1 + C_2 + 0 \\ 0 &= y'(0) = -C_1 - 2C_2 + 1 \end{aligned}$$

Solution: $C_1 = -1$ and $C_2 = 1$, so the forced solution is

$$y(t) = -e^{-t} + e^{-2t} + \frac{1}{2}\sin(2t)$$

(d) and (e): And Going...

- The transient part of the solution is $-e^{-t} + e^{-2t}$

  Solve $e^{-2t} - e^{-t} + 0.0001 = 0$

$$e^{-t} = \frac{1 \pm \sqrt{1 - 0.0004}}{2} \quad \Longrightarrow t \approx 9.210$$

- There is a limiting periodic solution: $\frac{1}{2}\sin(2t)$.

  The amplitude is $A = 1/2$ and the period is $T = \pi$.

Problem #2: Analyze a Spring-Mass System

A mass of 6kg is suspended from a spring that extended 0.1cm when a 1kg mass was attached. Pull the mass down 5cm and release it from rest.

(a) Basic Model: No damping, No forcing:

$$mx''(t) + kx(t) = 0, \quad x(0) = x_i,\ x'(0) = v_i.$$

(b) Determine some physical parameters...

(c) Add forcing...

(d) Add damping...

Analyze a Spring-Mass System

A mass of 6kg is suspended from a spring that extended 0.1cm when a 1kg mass was attached. Pull the mass down 5cm and release it from rest.

- Fix the parameters:

$$m = 6, \quad k = \frac{1 \cdot 9.8}{0.1} = 98, \quad x_i = -0.05, \quad v_i = 0$$

- Natural frequency $\omega_n = \sqrt{\dfrac{k}{m}} \approx 4.042$

  Period $T = \dfrac{2\pi}{\omega_n} \approx 1.555$

Keep Going ...

- Solve the IVP:

  $$\text{General Solution } C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

  Use the initial data to get $x(t) = -0.05 \cos(\omega_n t)$

- Amplitude $= 0.05$ meters.

- Make the damping critical:

  $\hookrightarrow p^2 \leq 4mk \implies p \approx 48.5$

System Behavior:

Solve the system

$$\vec{y}\,' = B \cdot \vec{y}$$

when

$$B := \begin{bmatrix} 11 & 2 \\ 1 & 10 \end{bmatrix}$$

A Little Algebra:

Eigenvalues:

$$\lambda_1 = \underline{\hspace{3cm}}, \quad \lambda_2 = \underline{\hspace{3cm}}$$

Eigenvectors:

$$\vec{v}_1 = \underline{\hspace{3cm}}, \quad \vec{v}_2 = \underline{\hspace{3cm}}$$

Sketch the Solution Curves:

System:

What Conditions on a, b, c, d guarantee that the homogeneous solution will be transient?

$$\begin{array}{rclcl} y' & = & ay & + & bz \\ z' & = & cy & + & dz \end{array}$$

CE: $r^2 - (a+d)r + (ad - bc) = 0$

Two roots: $r = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

These will be negative or have negative real part if

$$(a+d) < 0 \quad \text{and} \quad ad > bc$$

Announcements:

Project #2: Due Thursday at Noon in SH202A

Evaluation Time:

Special Request: Would calculators make a big difference on exams? Please explain why.