TOPICS

- General Solution Method: Separation of Variables
  - $\hookrightarrow$ Just Integrate
  - $\hookrightarrow$ Trouble? When?
- Solution Methods for Linear Equations
  - $\hookrightarrow$ Homogeneous + Particular
  - $\hookrightarrow$ Finding the Particular = Work

Example: Separable Equation

$$\frac{dP}{dt} = (a - sP)P$$

leads to

$$\int \frac{dP}{(a - sP)P} = \int dt$$

You can integrate both sides (really, you can!)

$$\frac{1}{(a - sP)P} = \frac{1/a}{a - sP} + \frac{1/a}{P}$$

so

$$\int \frac{dP}{(a - sP)P} = \frac{1}{a}\int \frac{dP}{a - sP} + \frac{1}{a}\int \frac{dP}{P}$$

The Dirty Work:

. . . and Finally

$$P(t) = \frac{a}{s - Ce^{-at}}$$

Note:

(1) $\lim_{t\to\infty} P(t) = \frac{a}{s-0} = \frac{a}{s}$

(2) This is NOT the General Solution.

("General Solution" is defined for LINEAR equations.)

(3) Does $\frac{1}{P}$ satisfy a linear equation?

More Examples:

- $y'(x) = y^2(x)\cos(x), \quad y(\pi) = 1/2$
- $\dfrac{dx}{dt} = tx^2(t), \quad x(0) = 1/3$
- $y'(x) = \ln(1 + y^2(x))e^{\cos(x)}, \quad y(100) = 6.02 \times 10^{23}$

Non-Separable:

- $\dfrac{dT}{dt} + \dfrac{Ak}{cm}T(t) = \dfrac{Ak}{cm}T_{\text{out}}(t)$

A Dirty Trick...

$$\frac{dy}{dx} + 3y = 2e^{-2x}$$

Recall: Linear Equations

The General Solution for

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

$$y(x) = y_p(x) + Cy_h(x)$$

where

(1) $y_p(x)$ is *any* solution for the ODE.

(2) $y_h(x)$ is a non-trivial solution for the corresponding homogeneous equation.

Example:

$\dfrac{dP}{dt} = kP - H$ has general solution

$$P(t) = \frac{H}{k} + Ce^{kt}$$

BUILD a Particular Solution

Find a *particular solution* for

$$\frac{dy}{dx} + g(x)y = f(x)$$

Key Idea: y_p has to be *different* from y_h.

$$y_p(x) = u(x) \cdot y_h(x)$$

Plug it in ...find an equation for $u(x)$

this is called "Variation of Parameters"

More V. of P.

Plug it in:

$$(u(x) \cdot y_h(x))' + g(x)(u(x) \cdot y_h(x)) = f(x)$$

Expand and combine:

$$u' y_h + u\left[y_h' + g y_h\right] = f(x)$$

Conclude:

$$u'(x) = \frac{f(x)}{y_h(x)} \implies u(x) = \int \frac{f(x)}{y_h(x)} dx$$

Summary: To Solve

$$y' + g(x)y = f(x)$$

(1) find the homogeneous solution $y_h(x)$:

$$y_h(x) = \exp\left(-\int g(x)dx\right)$$

(2) find the particular solution $y_p(x)$:

$$y_p(x) = u(x)y_h(x) = \left[\int \frac{f(x)}{y_h(x)}dx\right] y_h(x)$$

(3) combine the results:

$$y(x) = y_p(x) + Cy_h(x)$$

Examples:

Harvesting: $P'(t) = kP(t) - H$

Homogeneous Solution: $P_h(t) = e^{kt}$

Variation of Parameters:

$$u(t) = \int \frac{-H}{e^{kt}}dt = -H\int e^{-kt}dt = \frac{H}{k}e^{-kt}$$

Particular Solution: $P_p(t) = u(t)e^{kt} = \dfrac{H}{k}$

BUILD THE GENERAL SOLUTION

$$\frac{dT}{dt} + \frac{Ak}{cm}T(t) = \frac{Ak}{cm}T_{\text{out}}(t)$$

(1) Solve $\dfrac{dT}{dt} + \dfrac{Ak}{cm}T(t) = 0$

$$T_h(t) = e^{-\frac{Ak}{cm}t}$$

(2) Find a particular solution:

$$T_p(t) = ???$$

Fix Some Numbers:

$$\gamma = 0.3, \quad T_{\text{out}}(t) = 273 + 0.2t$$

Variation of Parameters:

$$T_p(t) = \int \frac{\gamma T_{\text{out}}(t)}{e^{-\gamma t}} dt = \int 0.3(273 + 0.2t)e^{0.3t} dt$$

Integration by parts:

$$T_p(t) = 272.3333 + 0.2t$$

Remarks:

$$\lim_{t \to \infty} T_h(t) = 0$$

$$T_p(t) = (273 + 0.2t) - 2/3 = T_{\text{out}}(t) - 2/3$$

Graph It...

HW #3: Due Monday in Lecture

Sec 2.3.1: 1a, 5(a,b,c), 9;

Sec 2.4.5: 11, 13;

Sec 4.1.6: 7–10, 38, 40;

+ Go to your favorite non-Math book and bring back an ODE.

You must provide:

(1) definitions of all variables

(2) *brief* explanation of application

(3) a complete reference (text and page)