TOPICS

- Second-Order Models

  $\hookrightarrow$ Springs and masses

  $\hookrightarrow$ Finding parameters

  $\hookrightarrow$ Initial date times 2

Springs and Masses

Hooke's Law:

$$\left[\text{ force exerted by a spring} = k \times \text{extension of the spring }\right]$$

Add Newton's Law and a coordinate system:

$$mx''(t) = -kx(t)$$

General Solution:

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

More Detail: Balance the Forces

- Force of Gravity: $F_{\mathrm{g}} = -mg$
- Spring Force: $F_{\mathrm{s}} = k \cdot (\text{stretch}) = k \cdot \Big[(h - x(t)) - (h - x_{\mathrm{u}})\Big]$

$$mx''(t) = F_{\mathrm{total}} = F_{\mathrm{s}} + F_{\mathrm{g}}$$

or

$$mx''(t) = k \cdot \Big[x_u - x(t)\Big] - mg$$

or

$$mx''(t) = -kx(t) + \Big[k \cdot x_u - mg\Big]$$

Example:

A 3kg mass stretches a spring 0.02 meters.

Suppose that a 2 kg mass is attached to the same spring and it is set in motion by pulling it down 5cm and releasing it from rest.

Write an initial value problem to model this situation.

Initial Data: <u>Position</u> and <u>Velocity</u>

A Little Qualitative Analysis

What does the model say?

$$mx''(t) = -kx(t)$$

- Increasing and Decreasing?
- Concave up and concave down?
- Steady States?

Characteristic Equations... Again

Look for solutions of the form $\sin(\omega t)$ and $\cos(\omega t)$:

$$m\Big(\sin(\omega t)\Big)'' + k\sin(\omega t) = 0$$

$$\left(-m\omega^2 + k\right)\sin(\omega t) = 0$$

Two roots:

$$\omega = \pm\sqrt{\frac{k}{m}}$$

Characteristic Equations... AGAIN!

Look for solutions of the form e^{rt} (just as for first-order).

Plug it in:

$$0 = m\left(e^{rt}\right)'' + \, ke^{rt} = \left(mr^2 + k\right)e^{rt}$$

Two roots:

$$r = \pm i\sqrt{\frac{k}{m}}$$

Add Friction Damping

<u>Assume</u>

- a friction force directed opposite the velocity vector
- magnitude of the force is proportional to the velocity

$$F_{\text{friction}} = -px'(t)$$

New Model:

$$mx''(t) + px'(t) + kx(t) = 0$$

Example

A force of 0.36 Newtons is required to drag a 7 kg mass across a table at a speed of 0.4 m/s.

$$0.36 = p \cdot (0.4) \Longrightarrow p = \frac{0.36}{0.4} = 0.9$$

Aside: Is this a coefficient of static or kinetic friction?

Simple Pendulum

Basic Trig:

If θ is measured in radians, then $L\theta$ is distance along the circumference of a circle.

$$\text{Acceleration} = (L\theta)'' = L\theta''$$

$\hookrightarrow$ Force of Gravity: $-mg\sin(\theta)$

$\hookrightarrow$ Model: $mL\theta'' = -mg\sin(\theta)$

Linearize the Pendulum

"Recall"

$$\sin(\theta) = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \cdots$$

$$L\theta'' = -g\sin(\theta) = g\theta - g\frac{1}{3!}\theta^3 + g\frac{1}{5!}\theta^5 + \cdots$$

Final Model

$$L\theta'' + g\theta = 0, \quad \theta(0) = \theta_i,\ \theta'(0) = \omega$$

Remarks

The solution will look like

$$\theta(t) = A\cos\left(\sqrt{\frac{g}{L}}\,t\right)$$

- What are the initial data?
- What is the period?
- What is the amplitude of the oscillation?

HW #7: Due Friday in Lecture

Sec 5.1: 1, 9, 12 ;

Sec 5.3: 5, 6 ;

Sec 5.4: 1 ;

Project Reports Due Tomorrow in Conference!