**TOPICS**

- CE for Constant Coefficients

  $\hookrightarrow$ Three Things Happen

- Finding a Particular Solution

  $\hookrightarrow$ UC still works



**Summary: To solve**

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1 y_1(x) + C_2 y_2(x)$$

**Test for Linear Independence**

Two solutions for

$$y'' + P(x)y' + Q(x)y = 0$$

are linearly independent on their domain if the Wronskian is not zero for at least one point in the interval.

**Example:**

Show that $y_1(x) = x^2$ and $y_2(x) = x^{-2}$ are linearly independent solutions for

$$x^2y'' + xy' - 4y = 0$$



**Remarks:**

- The Wronskian is a function of $x$.
- $W(x)$ satisfies its own ODE!

$$\frac{dW}{dx} + P(x)W(x) = 0$$

if $y_1$ and $y_2$ are solutions for

$$y'' + P(x)y' + Q(x)y = 0$$

## Characteristic Equations

The key to the constant coefficient universe!

To solve

$$mx'' + px' + kx = 0$$

assume that

$$x(t) = e^{rt}$$

and forge ahead:

$$\begin{aligned} mx'' + px' + kx &= m\left(e^{rt}\right)'' + p\left(e^{rt}\right)' + k\left(e^{rt}\right) \\ &= m\left(r^2 e^{rt}\right) + p\left(re^{rt}\right) + k\left(e^{rt}\right) \\ &= \left(mr^2 + pr + k\right)e^{rt} \end{aligned}$$



## Characteristic Equation

$$mr^2 + pr + k = 0$$

**Three Cases:**

- Two, Distinct Real Roots: $r_1$ and $r_2$
- One, Repeated Real Root: $r = r_1 = r_2$
- Two, Complex Conjugate Roots:
  $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$

**Distinct Roots $\Longrightarrow$ Life is Good**

- $y'' - 9y = 0 \implies r^2 - 9 = 0$

  $r_1 = +3$, $r_2 = -3$ so $y_1(x) = e^{3x}$ and $y_2(x) = e^{-3x}$

- $x'' + 3x' + 2x = 0 \implies r^2 + 3r + 2 = 0$

  $r_1 = -1$, $r_2 = -2$ so $x_1(t) = e^{-t}$ $x_2(t) = e^{-2t}$

- Find an equation with solutions

$$y_1(x) = e^{3x} \text{ and } y_2(x) = e^{-(1/2)x}$$



**Repeated Root: Life is Pretty Good**

- $y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = 0$

  $r_1 = -2$ and $r_2 = -2$ gives

  $y_1(x) = e^{-2x}$ and $y_2(x) = ???$

To build another solution, assume that

$$y_2(x) = u(x)y_1(x)$$

Plug it in and cancel away:

**Plugging...**

$$
\begin{aligned}
0 &= \Big(u(x)y_1\Big)'' + 4\Big(u(x)y_1\Big)' + 4\Big(u(x)y_1\Big) \\
&= \Big(u''y_1 + 2u'y_1' + uy_1''\Big) + 4\Big(u'y_1 + uy_1'\Big) + 4uy_1 \\
&= u''y_1(x) \\
&\qquad + u'\Big(2y_1' + 4y_1\Big) + u\Big(y_1'' + 4y_1' + 4y_1\Big)
\end{aligned}
$$

The whole thing reduces to

$$u''y_1(x) = 0$$



**Test Independence**

$$y_1(x) = e^{rx} \quad \text{and} \quad y_2(x) = xe^{rx} \quad \text{are L.I.D.}$$

Compute the Wronskian:

$$W(y_1, y_2) = \det \begin{bmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx} + rxe^{rx} \end{bmatrix}$$

You get

$$W(y_1, y_2) = e^{2rx} + rxe^{2rx} - rxe^{2rx} = e^{2rx}$$

$\star$ and this is *never* zero $\star$

**Complex Conjugate Roots:**

Standard, unforced, undamped spring model:

$$mx'' + kx = 0 \implies mr^2 + k = 0$$

$$r_1 = +i\sqrt{\frac{k}{m}} \quad \text{and} \quad r_2 = -i\sqrt{\frac{k}{m}}$$

**Example:** Solve

$$y'' + 2y' + 2y = 0$$

**CE:** $r^2 + 2r + 2 = 0$ has roots

$$r = \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm \frac{1}{2}\sqrt{-4} = -1 \pm i$$



**Continue the Example:**

You have L.I.D. solutions:

$$y_1(x) = e^{(-1+i)x} \quad \text{and} \quad y_2(x) = e^{(-1-i)x}$$

A more useful form:

$$e^{(-1+i)x} = e^{-x}e^{ix} = e^{-x}\Big[\cos(x) + i\sin(x)\Big]$$

$$e^{(-1-i)x} = e^{-x}e^{-ix} = e^{-x}\Big[\cos(x) - i\sin(x)\Big]$$

**Compare Two General Solutions**

$$y(x) = c_1 e^{(-1+i)x} + c_2 e^{(-1-i)x}$$

and

$$y(x) = e^{-x}\Big[A\sin(x) + B\cos(x)\Big]$$

**Questions:**

Are these the same solution?

How are the constants related?



**Example: Solve an IVP**

$$y''(x) + 2y'(x) + 2y(x) = 0, \quad y(0) = 10,\ y'(0) = 5$$

Solve for $A$ and $B$

$$10 = y(0) = e^0\Big[A\sin(0) + B\cos(0)\Big] = B$$

$$\begin{aligned} 5 &= y'(0) \\ &= -e^0\Big[A\sin(0) + B\cos(0)\Big] + e^{-0}\Big[A\cos(0) - B\sin(0)\Big] \\ &= -10 + A \end{aligned}$$

So: $A = 15$ and $B = 10$

**Remarks:**

- Used

$$y'(x) = -e^{-x}\left[A\sin(x) + B\cos(x)\right] + e^{-x}\left[A\cos(x) - B\sin(x)\right]$$

- $A$ and $B$ are Real
- $c_1$ and $c_2$ are *complex*

$$c_1 = 5 + \frac{15}{2}i, \quad c_2 = 5 - \frac{15}{2}i$$



**HW #8: Due Tuesday in Conference**

**Sec 6.1: 11–13** ;

**Sec 6.2: 5, 10** ;

**Sec 6.3: 1, 3, 5** ;

**Project are being reviewed**