TOPICS

- Undetermined Coefficients
  - ↪ Constant Coefficients
  - ↪ Special Forcing Terms
- Numerical Methods
  - ↪ Euler is Back!

Summary: To solve

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x)$$

(1) find two linearly independent homogeneous solutions

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

(2) find the particular solution $y_p(x)$:

(3) combine the results:

$$y(x) = y_p(x) + C_1 y_1(x) + C_2 y_2(x)$$

Example:

Solve

$$x''(t) + 4x(t) = \sin(t)$$

- Step 1: $x_h(t) = c_1 \sin(2t) + c_2 \cos(2t)$

  $\hookrightarrow$ **C.E.** $r^2 + 4 = 0 \implies r = \pm 2i$

- Step 2: Find a particular solution:

  Try $x_p(t) = A\sin(t) + B\cos(t)$

  Plug it in an solve for A and B.

Dirty Work:

- Step 2: Solve for A and B:

$$\Big[-A\sin(t) - B\cos(t)\Big] + 4\Big[A\sin(t) + B\cos(t)\Big] = \sin(t)$$

Two equations:

$$-A + 4A = 1, \quad -B + 4B = 0$$

$$A = \frac{1}{3}, \quad B = 0$$

- General Solution:

$$x(t) = \frac{1}{3}\sin(t) + C_1 \sin(2t) + C_2 \cos(2t)$$

Undetermined Coefficients Works

- $f(x) = e^{\lambda x}$ ..... $y_p(x) = Ae^{\lambda x}$
- $f(x) = \sin(\omega t)$ ..... $y_p(x) = A\sin(\omega t) + B\cos(\omega t)$
- $f(x) = x^n$ ..... $y_p(x) = A_0 + A_1 x^1 + A_2 x^2 + \cdots + A_n x^n$

$\hookrightarrow$ **Remarks:**

$\star$ Watch for homogeneous solutions!

$\star$ It works for products too!:

$$f(x) = e^{2x}\sin(5x) \implies y_p(x) = e^{2x}\Big[A\sin(5x) + B\cos(5x)\Big]$$

Example: Exponentials All Around

Solve $y'' - 9y = e^x, \quad y(0) = 0,\ y'(0) = 0$

- $y_h(x) = C_1 e^{3x} + C_2 e^{-3x}$
- $y_p(x) = Ae^x$ leads to $A = -\frac{1}{8}$
- General Solution: $y(x) = -\dfrac{1}{8}e^x + C_1 e^{3x} + C_2 e^{-3x}$
- Use the initial data to find C_1 and C_2.

$$C_1 = \frac{1}{12}, \quad C_2 = \frac{1}{24}$$

Repeated Root: Life is Pretty Good

Solve $y'' + 4y' + 4y = 5e^{-2x}$

- CE gives $r_1 = -2$ and $r_2 = -2$ so

  $y_h(x) = C_1 e^{-2x} + C_2 x e^{-2x}$

- UC says to look for $y_p(x) =$???

  $y_p(x) = Ae^{-2x}$ won't work.

  $y_p(x) = Axe^{-2x}$ won't work.

  $y_p(x) = Ax^2 e^{-2x}$ will work.

The Dirty Work:

Guess $y_p(x) = Ax^2 e^{-2x}$ and plug it in

$$\begin{aligned} f(x) = 5e^{-2x} &= \left[2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2 e^{-2x}\right] \\ &\quad +4\left[2Axe^{-2x} - 2Ax^2 e^{-2x}\right] \\ &\quad +4\left[Ax^2 e^{-2x}\right] \\ &= 2Ae^{-2x} \\ &\quad +\left[-8Axe^{-2x} + 8Axe^{-2x}\right] \\ &\quad +\left[4Ax^2 e^{-2x} - 8Ax^2 e^{-2x} + 4Ax^2 e^{-2x}\right] \end{aligned}$$

Summary:

The general solution for $y'' + 4y' + 4y = 5e^{-2x}$

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \frac{5}{2} x^2 e^{-2x}$$

Questions:

- Can you choose the initial data so that the final solution contains only the particular solution?
- Does the solution have a transient component?
- Can you choose the initial data so that the final solution contains only the homogeneous solution?

Minute Quiz:

Find general solution for $y'' + 4y' + 4y = \left(5 \times 10^{23}\right) e^{-2x}$

Answer

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + \left[10^{23} \times\right] \frac{5}{2} x^2 e^{-2x}$$

Aside: Look at One Maple Solution

$$x''(t) + 4x(t) = \sin(2t)$$

$$\begin{aligned} x(\,t\,) = {} & \frac{1}{2}\cos(\,2\,t\,)\sin(\,t\,)\cos(\,t\,)^3 - \frac{1}{4}\cos(\,2\,t\,)\cos(\,t\,)\sin(\,t\,) \\ & - \frac{1}{4}\cos(\,2\,t\,)\arctan\left(\frac{\sin(\,t\,)}{\cos(\,t\,)}\right) + \frac{1}{2}\cos(\,t\,)^2\sin(\,2\,t\,) \\ & - \frac{1}{2}\cos(\,t\,)^4\sin(\,2\,t\,) + _C1\cos(\,2\,t\,) + _C2\sin(\,2\,t\,) \end{aligned}$$

Simplified a bit:

$$\begin{aligned} x(\,t\,) = {} & \frac{1}{8}\sin(\,2\,t\,) - \frac{1}{4}\cos(\,2\,t\,)\arctan\left(\frac{\sin(\,t\,)}{\cos(\,t\,)}\right) \\ & + _C1\cos(\,2\,t\,) + _C2\sin(\,2\,t\,) \end{aligned}$$

Euler's Method for Second-Order Equations

To approximate the solution to

$$mx''(t) + px'(t) + kx(t) = f(t), \quad x(0) = x_i,\ x'(0) = v_i$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= \frac{1}{m}f(t) - \frac{p}{m}v(t) - \frac{k}{m}x(t) \end{aligned}$$

Use Euler as before:

$$\begin{aligned} x(t+h) &= x(t) + v(t)\cdot h \\ v(t+h) &= v(t) + \left[\frac{1}{m}f(t) - \frac{p}{m}v(t) - \frac{k}{m}x(t)\right]\cdot h \end{aligned}$$

Example:

To approximate the solution to

$$x''(t) + 4x(t) = 0, \quad x(0) = 1,\ x'(0) = 0$$

$\star$ Turn the equation into a system:

$$\begin{aligned} x'(t) &= v(t), \quad x(0) = 1 \\ v'(t) &= -4x(t), \quad v(0) = 0 \end{aligned}$$

And Approximate:

$$\begin{aligned} x(t+h) &= x(t) + v(t) \cdot h, \quad x(0) = 1 \\ v(t+h) &= v(t) - 4x(t) \cdot h, \quad v(0) = 0 \end{aligned}$$

Take a Few Steps:

$$\begin{aligned} x_{n+1} &= x_n + v_n \cdot h, \quad x_0 = 1 \\ v_{n+1} &= v_n - 4x_n \cdot h, \quad v_0 = 0 \end{aligned}$$

n	0	1	2	3	4	5
t	0	0.10	0.20	0.30	0.40	0.50
x	1	1.00	0.96	0.88	0.76	0.68
v	0	−0.40	−0.80	−1.18	−1.54	−1.84

HW #8: Due Tomorrow in Conference

Sec 6.1: 11–13 ;

Sec 6.2: 5, 10 ;

Sec 6.3: 1, 3, 5 ;

HW #9: Due Friday in Lecture

Sec 6.4: 4, 17, 20;

Sec 6.5: 2, 9, 17 ;

Sec 6.6: 6 ;